Microscopic models of topological order

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(2023, CalTech, Burke Institute Lecture)

Phases of quantum matter ($T = 0$ phases)

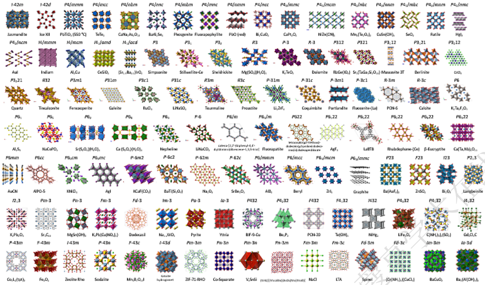
For a long time, we thought that Landau symmetry breaking classify all phases of matter

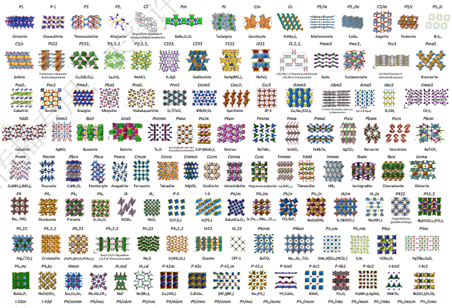
- **Symm. breaking phases are classified by a pair** $G_\Psi \subset G_H$

  G_H = symmetry group of the system (Hamiltonian).

  G_Ψ = symmetry group of the ground states.

- **230 crystals** from group theory





Can symm. breaking describe all phases of matter?

A spin-liquid theory of high T_c superconductors:


- 2d spin liquid $\to$ spin-charge separation:
  electron $\to$ *holon* $\otimes$ *spinon*,
  holon: charge-1 spin-0 boson,
  spinon: charge-0 spin-1/2 fermion.
  Holon condensation $\to$ high T_c superconductivity.
- **But how to characterize a spin liquid?**

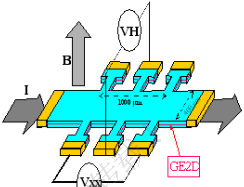
  One of the spin liquid is a state that break time reversal and parity symmetry, but not spin rotation symmetry, with order parameter $\boldsymbol{S}_1 \cdot (\boldsymbol{S}_2 \times \boldsymbol{S}_3) \neq 0$.
  $\to$ **Chiral spin liquid** Wen, Wilczek, Zee, PRB **39** 11413 (89)
- But we discovered several different chiral spin states with the same symmetry breaking.

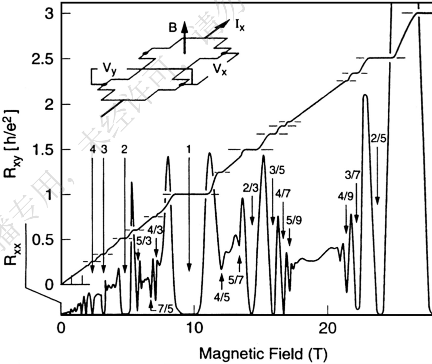
Topological orders in quantum Hall effect

- Quantum Hall states $R_{xy} = V_y / I_x = \frac{m}{n} \frac{2\pi\hbar}{e^2}$

  von Klitzing Dorda Pepper, PRL **45** 494 (1980)

  Tsui Stormer Gossard, PRL **48** 1559 (1982)





- FQH states have different phases even when there is no symm. and no symm. breaking.

- Chiral spin and FQH liquids must contain a new kind of order, named as **topological order**

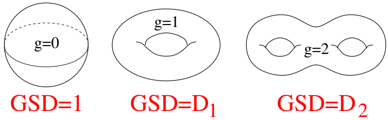
  Wen, PRB **40** 7387 (89); IJMP **4** 239 (90)

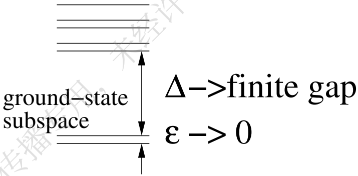
How to characterize chiral-spin/FQH liquids?

- How to extract universal information (topological invariants) from complicated many-body wave function $\Psi(x_1, \cdots, x_{10^{20}})$

  Put the gapped system on space with various topologies, and measure the ground state degeneracy.

  $\rightarrow$ The notion of **topological order**





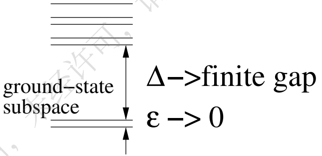
Wen PRB **40** 7387 (89)

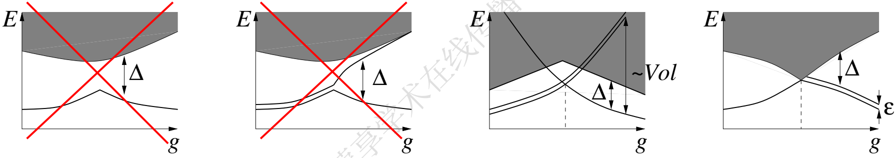
The ground state degeneracy is topological

- The ground state degeneracies are robust against any local perturbations that can break any symmetries. The ground state degeneracies have nothing to do with symmetry.
  $\rightarrow$ **topological degeneracy**

Wen Niu PRB **41** 9377 (90)

- The ground state degeneracies can only vary by some large changes of Hamiltonian
  $\rightarrow$ gap-closing phase transition.





What is the origin of topological degeneracy?

What is the origin and mechanism of topological order.

At first, the topological degeneracy was shown formally in using the so called large gauge transformation in gauge field theory.

Wen PRB **40** 7387 (89)

- In 2005, we discovered
  topological entanglement entropy
  Kitaev-Preskill hep-th/0510092
  Levin-Wen cond-mat/0510613



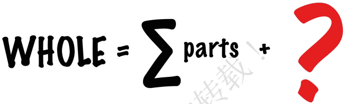
  and **long range quantum entanglement**
  Chen-Gu-Wen arXiv:1004.3835


- **Topological degeneracy (and topological order) comes from long range entanglement**

Why entanglement and degeneracy are related?

- For a highly entangled many-body quantum systems:
  knowing every parts still cannot determine the whole

\- In other words, there are different "wholes", that their every local parts are identical.



\- Local perturbations can only see the parts → those different "wholes" (the whole quantum states) have the same energy.

- What is "whole"?, what is "part"?
  whole = the whole wave function $|\Psi_\pm\rangle = \frac{|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle}{\sqrt{2}} = \Psi_\pm(m_1, m_2)$
  or the whole density matrix

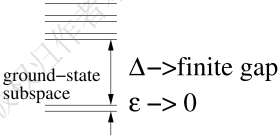
$$\rho^\pm = |\Psi_\pm\rangle\langle\Psi_\pm|, \qquad \rho^\pm_{m_1 m_2,\, m_1' m_2'} = \Psi^*_\pm(m_1, m_2)\Psi_\pm(m_1', m_2')$$

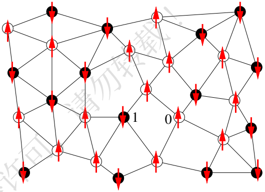
part = entanglement density matrix: $\rho_{\text{part}} = \mathrm{Tr}_{\text{other part}}(\rho_{\text{whole}})$

$$\rho^{\text{part-1}}_{m_1, m_1'} = \sum_{m_2} \rho^\pm_{m_1 m_2,\, m_1' m_2} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \langle H_{\text{part-1}}\rangle = \mathrm{Tr}(H_{\text{part-1}}\rho_{\text{part-1}})$$

Topological order as long range entanglement

- A **local quantum system** is described by $(\mathcal{V}_N, H_N)$
  $\mathcal{V}_N$: a Hilbert space with a tensor structure $\mathcal{V}_N = \otimes_{i=1}^N \mathcal{V}_i$
  H_N: a local Hamiltonian acting on $\mathcal{V}_N$:
  $H_N = \sum \hat{O}_{ij}$





\- *A ground state is not a single state in $\mathcal{V}_N$, but a subspace $\mathcal{V}_{grnd\ space} \subset \mathcal{V}_N$.*

- A **gapped quantum system** (a concept for $N \to \infty$ limit):
  $\{(\mathcal{V}_{N_1}, H_{N_1}); (\mathcal{V}_{N_2}, H_{N_2}); (\mathcal{V}_{N_3}, H_{N_3}); \cdots\}$ with gapped spectrum.

\- *A gapped quantum system is not a single Hamiltonian, but a sequence of Hamiltonian with larger and larger sizes.*

Local unitary trans. & gapped quantum phases

- Two gapped states, $|\Psi(0)\rangle$ and $|\Psi(1)\rangle$ (or more precisely, two gapped ground state subspaces), are in the same phase iff they are related through a local unitary (LU) evolution

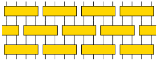
$$|\Psi(1)\rangle = P\left(e^{-\mathrm{i}\int_0^1 dg'\, H(g')}\right)|\Psi(0)\rangle$$

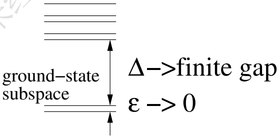
where $H(g) = \sum_i O_i(g)$ and $O_i(g)$ are local hermitian operators.

- $|\Psi(0)\rangle$ and $|\Psi(1)\rangle$ can be smoothly connect without closing the gap. Hastings-Wen cond-mat/0503554; Bravyi-Hastings-Michalakis arXiv:1001.0344
- LU evolution = *local unitary transformation*:

$$|\Psi(1)\rangle = P\left(\mathrm{e}^{-\mathrm{i}\,T\int_0^1 dg\, H(g)}\right)|\Psi(0)\rangle$$

$$= \left(\prod_{\Delta\tau} \mathrm{e}^{-\Delta\tau H}\right)|\Psi(0)\rangle$$

$$= \left(\prod_{\Delta\tau} \mathrm{e}^{-\Delta\tau H_A}\mathrm{e}^{-\Delta\tau H_B}\right)|\Psi(0)\rangle =$$  $|\Psi(0)\rangle$



where $H_A = \sum_{i\in A} O_i$, $H_B = \sum_{i\in B} O_i$

Local unitary trans. & gapped quantum phases

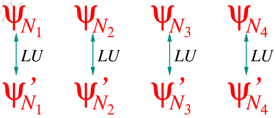
- The local unitary transformations define an equivalence relation: *Two gapped states related by a LU transition are in the same phase.*

  A gapped quantum phase is, by definition, an equivalence class of local unitary transformations Chen-Gu-Wen, arXiv:1004.3835.

  A gapped quantum phase:

  $H_{N_1}, H_{N_2}, H_{N_3}, H_{N_4}, \cdots$

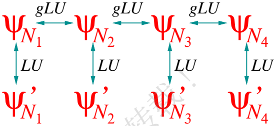
  $H'_{N_1}, H'_{N_2}, H'_{N_3}, H'_{N_4}, \cdots$



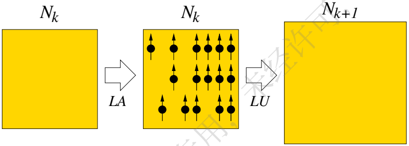
  OK definition only for translation invariant systems.

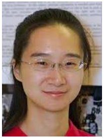
Gapped quantum liquid phase

- **A gapped quantum liquid phase:**
  $H_{N_1}, H_{N_2}, H_{N_3}, H_{N_4}, \cdots$
  $H'_{N_1}, H'_{N_2}, H'_{N_3}, H'_{N_4}, \cdots$
  $N_{k+1} = sN_k,\ \ s \sim 2$



- $\Psi_{N_{i+1}} \overset{LA}{\sim} \Psi_{N_i} \otimes \Psi^{dp}_{N_{i+1}-N_i}$. Generalized local unitary (gLU) trans.
  where
  $\Psi^{dp}_N = \otimes^N_{i=1} |\uparrow\rangle$





- gLU transformations allow us to take the thermal dynamical limit ($N_k \to \infty$ limit) without translation symmetry.

Zeng-Wen, arXiv:1406.5090

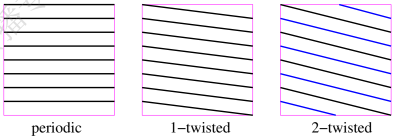
- **Gapped quantum liquid phase = Topological order**

Examples and beyond gapped quantum liquid

Different gapped quantum systems are described by different sequences of Hamiltonians $\{H_{N_k}\}$:

- Trivial gapped quantum liquid: $|\Psi_{\text{ground}}\rangle = |\cdots\uparrow\uparrow\uparrow\cdots\rangle$ from $H_{N_k}^{\text{trivial-liquid}} = -\sum_{i=1}^{N_k} Z_i$
- Transverse Ising model in symmetry breaking phase $\rightarrow$ a gaped quantum liquid. Ground state degeneracy $GSD = 2$
- Stacking 2+1D topologically ordered states with ground state degeneracy $m \neq 1$ on torus $\rightarrow$ a gapped quantum state, but not a gapped quantum liquid.

\- Layered topological order:
Ground state degeneracy can be $GSD = m^{L_z}, m, m^2$

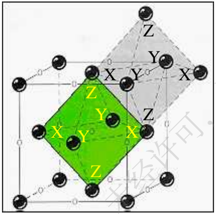


Fracton state – commuting operator models

- Chamon's **quantum-glass** model on 3D FCC lattice:

Chamon, cond-mat/0404182

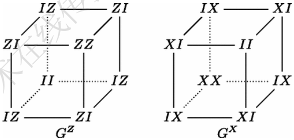
$$H = -\sum_{cubes} O_{\boldsymbol{I}} + O_{\boldsymbol{I}+\frac{1}{2}\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}} + O_{\boldsymbol{I}+\frac{1}{2}\boldsymbol{x}+\frac{1}{2}\boldsymbol{z}} + O_{\boldsymbol{I}+\frac{1}{2}\boldsymbol{y}+\frac{1}{2}\boldsymbol{z}}$$

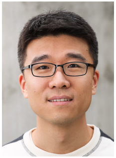


- Haah's **cubic code** on 3D cubic lattice:

Haah, arXiv:1101.1962

$$H = -\sum_{cubes} (G^Z + G^X),$$





More exotic long-range entanglement

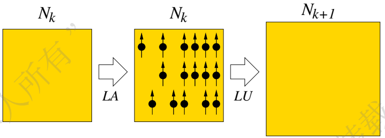
- Topo. order = gapped quantum **liquid** Zeng-Wen14; Swingle-McGreevy14

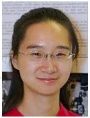
  $LU|\Psi_{N_{k+1}}\rangle = |\Psi_{N_k}\rangle \otimes |\Psi^{\text{prod}}\rangle$

  → gauge theory

  → quantum field theory

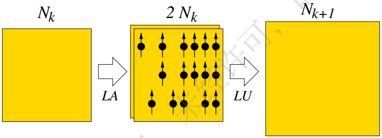
  → MERA rep. Vidal 06





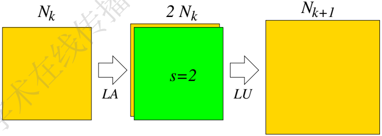
- **s-source**: $LU|\Psi^{s=2}_{N_{k+1}}\rangle = |\Psi_{N_k}\rangle \otimes |\Psi_{N_k}\rangle \otimes |\Psi^{\text{prod}}\rangle$ Swingle-McGreevy 14

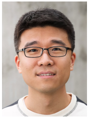
- Quantum liquid has $s = 1$
- 3D layered FQH: $s = 2$
- d+1D Fermi liquid: $s = \frac{2^d}{2}$
- no MERA rep.



- Haah's cubic code: $LU|\Psi_{N_{k+1}}\rangle = |\Psi_{N_k}\rangle \otimes |\Psi^{s=2}_{N_k}\rangle \otimes |\Psi^{\text{prod}}\rangle$ Haah 11

- no MERA rep.
- No quantum field theory description





Many-body entanglement goes beyond quantum field theory.

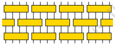
Topo. order = Pattern of long range entanglement

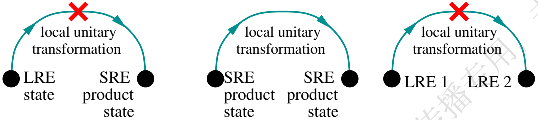
Chen-Gu-Wen arXiv:1004.3835

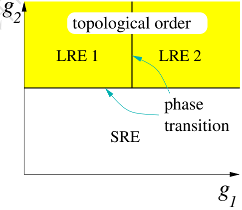

For gapped systems with no symmetry:

- According to Landau theory, no symm. to break $\to$ all systems belong to one trivial phase
- Thinking about entanglement: there are

- **long range entangled (LRE) states** $\to$ many phases
- **short range entangled (SRE) states** $\to$ one phase

$|\text{LRE}\rangle \neq$  $|\text{product state}\rangle = |\text{SRE}\rangle$





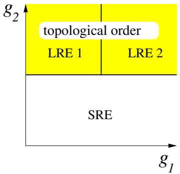
- All SRE states belong to the same trivial phase
- LRE states can belong to many different phases: different patterns of long-range entanglements defined by LU trans. = different **topological orders** Wen, Phys. Rev. B40, 7387 (1989)

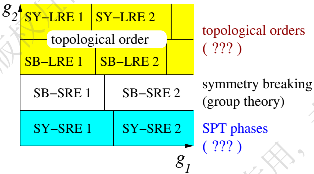
Gapped liquid phase with symm: SET/SPT phase

For gapped systems with a symmetry $H = U_g H U_g^\dagger,\ g \in G$

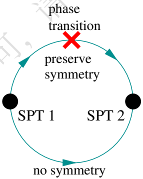
- **LRE symmetric states** → many different phases
- **SRE symmetric states** → one phase (no symm. breaking)

We may call them **symm. protected trivial (SPT)** phase



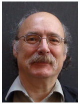
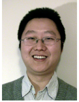


Gu Wen arXiv:0903.1069



- SPT phases = equivalent class of *symmetric* LU trans.
- Examples: 1D spin-1 gapped phase Haldane 83; Gu-Wen 09,

2D TI Kane-Mele 05; Bernevig-Zhang 06, 3D TI Moore-Balents 07; Fu-Kane-Mele 07

1D  2D     3D   

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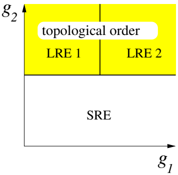

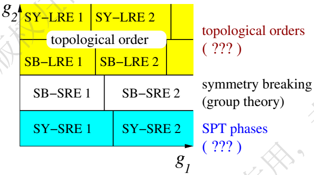

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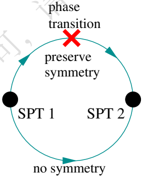

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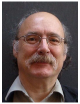
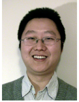
1D  2D     3D   

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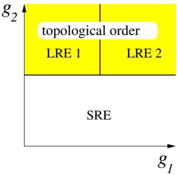
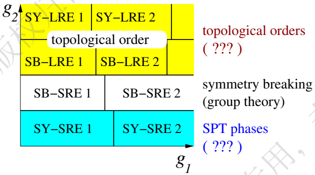
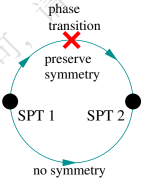
We may call them **symm. protected trivial (SPT)** phase
or **symm. protected topological (SPT)** phase

Gu Wen arXiv:0903.1069







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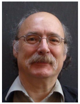
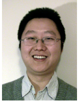
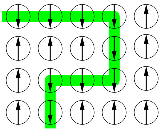
- Examples: 1D spin-1 gapped phase Haldane 83; Gu-Wen 09,
  2D TI Kane-Mele 05; Bernevig-Zhang 06, 3D TI Moore-Balents 07; Fu-Kane-Mele 07

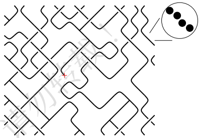
1D  2D     3D   

Example of Topologically ordered states

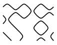
To make topological order, we need to sum over many different product states, but we should not sum over everything.

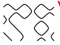
$\sum_{\text{all spin config.}} |\uparrow\downarrow ..\rangle = |\rightarrow\rightarrow ..\rangle$

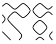


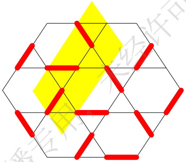


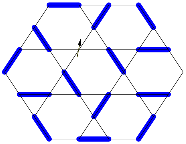
- *sum* over a subset of spin config.:

$|\Phi^{Z_2}_{\text{loops}}\rangle = \sum \Big|$  $\Big\rangle$

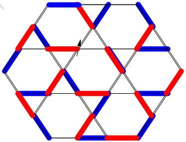
$|\Phi^{DS}_{\text{loops}}\rangle = \sum (-)^{\#\text{ of loops}} \Big|$  $\Big\rangle$

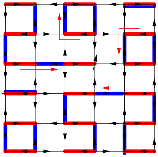
$|\Phi^{\theta}_{\text{loops}}\rangle = \sum (e^{i\theta})^{\#\text{ of loops}} \Big|$  $\Big\rangle$



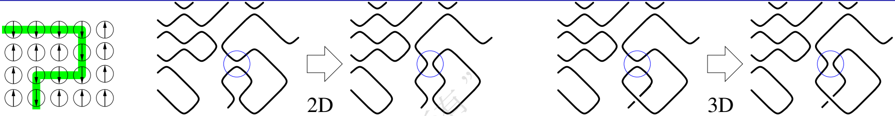


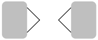
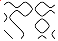
- Can the above wavefunction be the ground states of local Hamiltonians?





Local rule → global wave function



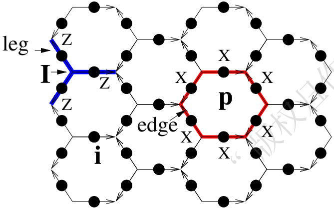
- Local rules of a string liquid:
  (1) Dance while holding hands (no open ends)
  (2) $\Phi_{\text{str}}\Big($$\Big) = \Phi_{\text{str}}\Big($$\Big)$, $\Phi_{\text{str}}\Big($$\Big) = \Phi_{\text{str}}\Big($$\Big)$
  → Global wave function $\Phi_{\text{str}}\Big($$\Big) = 1$
- Local rules of another string liquid:
  (1) Dance while holding hands (no open ends)
  (2) $\Phi_{\text{str}}\Big($$\Big) = \Phi_{\text{str}}\Big($$\Big)$, $\Phi_{\text{str}}\Big($$\Big) = -\Phi_{\text{str}}\Big($$\Big)$
  → Global wave function $\Phi_{\text{str}}\Big($$\Big) = (-)^{\#\text{ of loops}}$
- Two topo. orders: Z_2 **topo. order** Read-Sachdev PRL 66, 1773 (91), Wen PRB 44, 2664 (91), Moessner-Sondhi PRL 86 1881 (01) and **double-semion topo. order** Freedman etal cond-mat/0307511, Levin-Wen cond-mat/0404617

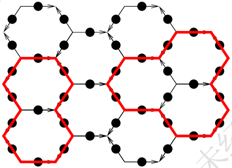
Toric-code model – Z_2 topological order

Local rule $\Phi_{str}(\ldots) - \Phi_{str}(\ldots) = \Phi_{str}(\ldots) - \Phi_{str}(\ldots) = 0$

$\rightarrow$ local Hamiltonian $\hat{P}\Phi_{str} = 0$.

- The Hamiltonian to enforce the local rules: Kitaev quant-ph/9707021





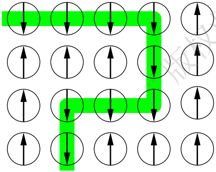
$$H = -U\sum_{I}\hat{Q}_I - g\sum_{p}\hat{F}_p, \quad \hat{Q}_I = \prod_{\text{legs of } I} Z_i, \quad \hat{F}_p = \prod_{\text{edges of } p} X_i$$

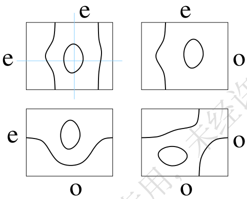
- The Hamiltonian is a sum of commuting operators
$[\hat{F}_p, \hat{F}_{p'}] = 0$, $[\hat{Q}_I, \hat{Q}_{I'}] = 0$, $[\hat{F}_p, \hat{Q}_I] = 0$. $\hat{F}_p^2 = \hat{Q}_I^2 = 1$

- Ground state $|\Psi_{grnd}\rangle$: $\hat{F}_p|\Psi_{grnd}\rangle = \hat{Q}_I|\Psi_{grnd}\rangle = |\Psi_{grnd}\rangle$
$\rightarrow (1-\hat{Q}_I)\Phi_{grnd} = (1-\hat{F}_p)\Phi_{grnd} = 0$.

Many-body energy spectrum of toric code model

- The $-U\sum_I \hat{Q}_I$ enforce closed-string ground state.
- $\hat{F}_p$ adds a small loop and generates a permutation among the loop states $\left| \text{loops} \right\rangle \to$ Ground states on torus $|\Psi^{\alpha}_{\text{grnd}}\rangle = \sum_{\text{loops}} \left| \text{loops} \right\rangle$
- There are four degenerate ground states $\alpha = ee, eo, oe, oo$



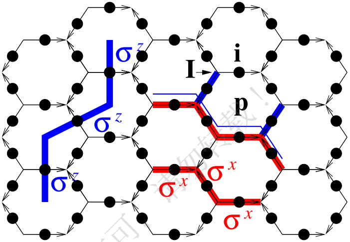


$D^{\text{tor}}=4$

- On genus g surface, ground state degeneracy $D_g = 4^g$
- Quasiparticle excitation energy gap (from thermal activation)
  $\Delta_p^Q = 2U$, $\Delta_p^F = 2g$
  Spectrum energy gap (from neutron scattering, ...)
  $\Delta^Q = 4U$, $\Delta^F = 4g$ (Double the quasiparticle gap)

String operators that create topological excitations

- Toric code model:
  $H = -U \sum_I \hat{Q}_I - g \sum_p \hat{F}_p$
  $\hat{Q}_I = \prod_{\text{legs of } I} Z_i$,
  $\hat{F}_p = \prod_{\text{edges of } p} X_i$



- Topological excitations: flipping $\hat{Q}_I, \hat{F}_p$
  e-type: $\hat{Q}_I = 1 \to \hat{Q}_I = -1$
  m-type: $\hat{F}_p = 1 \to \hat{F}_p = -1$
- Type-e string operator $W_e = \prod_{\text{string}} X_i$
- Type-m string operator $W_m = \prod_{\text{string}^*} Z_i$
- Type-f string op. $W_f = \prod_{\text{string}} X_i \prod_{\text{legs}} Z_i$
- $[H, W_e^{\text{loop}}] = [H, W_m^{\text{loop}}] = 0$. $\to$ Closed strings cost no energy
- $[\hat{Q}_I, W_e^{\text{open}}] \neq 0$ flip $\hat{Q}_I \to -\hat{Q}_I$, $[\hat{F}_p, W_m^{\text{open}}] \neq 0$ flip $\hat{F}_p \to -\hat{F}_p$
  $\to$ open-string create a pair of topo. excitations at their ends.
- **Fusion algebra** of string operators $\to$ fusion of topo. excitations:
  $W_e^2 = W_m^2 = W_f^2 = W_e W_m W_f = 1$ when strings are parallel

String operators that create topological excitations

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  $H = -U \sum_I \hat{Q}_I - g \sum_p \hat{F}_p$
  $\hat{Q}_I = \prod_{\text{legs of } I} Z_i,$
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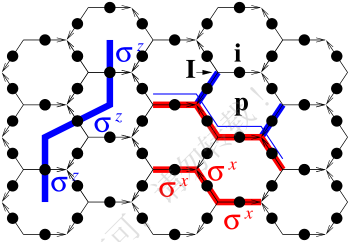

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  $\hat{Q}_I = \prod_{\text{legs of } I} Z_i$,

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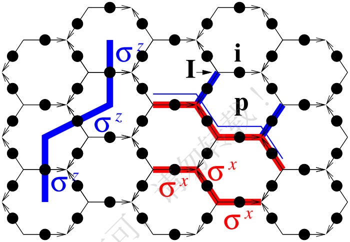

- Topological excitations: flipping $\hat{Q}_I, \hat{F}_p$

  e-type: $\hat{Q}_I = 1 \to \hat{Q}_I = -1$

  m-type: $\hat{F}_p = 1 \to \hat{F}_p = -1$
- Type-e string operator $W_e = \prod_{\text{string}} X_i$ $\to$ e-type. $e \otimes e = \mathbf{1}$
- Type-m string operator $W_m = \prod_{\text{string}^*} Z_i$ $\to$ m-type. $m \otimes m = \mathbf{1}$
- Type-f string op. $W_f = \prod_{\text{string}} X_i \prod_{\text{legs}} Z_i$ $\to$ f-type $= e \otimes m$
- $[H, W_e^{\text{loop}}] = [H, W_m^{\text{loop}}] = 0$. $\to$ Closed strings cost no energy
- $[\hat{Q}_I, W_e^{\text{open}}] \neq 0$ flip $\hat{Q}_I \to -\hat{Q}_I$, $[\hat{F}_p, W_m^{\text{open}}] \neq 0$ flip $\hat{F}_p \to -\hat{F}_p$

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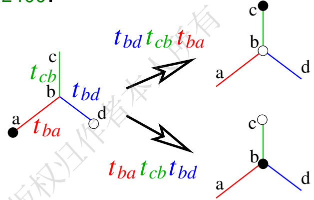
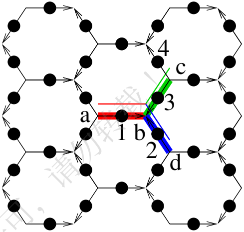
Statistics of ends of strings

- The statistics is determined by particle hopping operators

Levin-Wen cond-mat/0302460:





- An open string operator is a hopping operator of the 'ends'. The algebra of the open string operator determine the statistics.
- For type-e string: $t_{ba} = X_1$, $t_{cb} = X_3$, $t_{bd} = X_2$
  We find $t_{bd}t_{cb}t_{ba} = t_{ba}t_{cb}t_{bd}$
  The ends of type-*e* string are bosons
- For type-f strings: $t_{ba} = X_1$, $t_{cb} = \underline{X_3}Z_4$, $t_{bd} = X_2\underline{Z_3}$
  We find $t_{bd}t_{cb}t_{ba} = -t_{ba}t_{cb}t_{bd}$
  The ends of type-*f* strings are fermions

Topological excitations and higher symmetry

- Point-like topological excitations are created by string operators. String-like topo. excitations are created by membrane operators. The interior of string/membrane operators commute with all the local terms in the Hamiltonian and does not create any energy. The boundary of string/membrane operators does not commute the Hamiltonian, and create energy that corresponds to the topological excitation.

  The existence of topological excitations implies higher symmetry → logical operators in topo. quantum computation

- The toric code model has a **higher symmetry**:

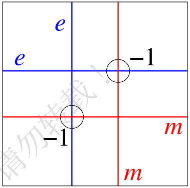
$$[H, W_e^{\text{loop}}] = [H, W_m^{\text{loop}}] = 0, \quad \text{for any closed loops}$$

The symmetry, whose transformations are generated by all closed co-dimension-1 extended operators, is called a 1-symmetry.

Nussinov Ortiz, arXiv:cond-mat/0702377 (called gauge-like symmetry)

Gaiotto Kapustin Seiberg Willett, arXiv:1412.5148 (called higher-form symmetry)

Topological ground state degeneracy and spontaneous higher symmetry breaking



- When strings cross, $W_e W_m = (-)^{\#\text{ of cross}} W_m W_e$
  → 4^g degeneracy on genus g surface
  → **Topological degneracy**
  Degeneracy remain exact for any perturbations localized in a finite region.
- The ground state degeneracy remain exact even if we explicitely break the 1-symmetry on lattice.

This hints a general result: Pace Wen arXiv:2301xxxxx

Emergent higher symmetry is always exact, while emergent 0-symmetry is not exact.

- Breaking of emergent 0-symmetry $\sim E^{\gamma} + L^{-\nu}|_{L\to\infty} \to$ non-zero.
- Breaking of emergent higher symmetry $\sim \mathrm{e}^{-L^{\nu}}|_{L\to\infty} \to 0$.

This makes higher symmetry useful in condensed matter.

Foerster Nielsen Ninomiya, Phys. Lett. B **94**, 135 (80); Hastings Wen cond-mat/0503554

Topological ground state degeneracy and spontaneous higher symmetry breaking

- The ground states of toric code model spontaneously break the 1-symmetry.

  Def: A (higher/algebraic/non-invertible) symmetry is spontaneously broken if the symmetry transformations is not $\propto$ id in the ground state subspace, for some closed spaces.

  A (higher/algebraic/non-invertible) symmetry is not spontaneously broken if the ground state not degenerate for all closed spaces.

- In the presence of the 1-symmetry, the ground state degeneracy (and topological order) can be viewed as coming from the **spontaneous breaking of the 1-symmetry**.

- In the absence of the 1-symmetry, the ground state degeneracy (and topological order) can be viewed as coming from the **spontaneous breaking of the exact emergent 1-symmetry**.

A closer look at higher symmetry and anomaly

- The toric model has two 1-symmetries $\mathbb{Z}_2^{(1)e}$ and $\mathbb{Z}_2^{(1)m}$, generated by two loop operators $W_e^{\text{loop}}, W_m^{\text{loop}}$.

  But the combined symmetry is not $\mathbb{Z}_2^{(1)e} \times \mathbb{Z}_2^{(1)m}$.
  It is $\mathbb{Z}_2^{(1)e} \vee \mathbb{Z}_2^{(1)m}$ – a twisted product, because the symmetry transformations of the two symmetries do not commute:
  $W_e^{\text{loop}} W_m^{\text{loop}} = (-)^{\#\text{ of cross}} W_m^{\text{loop}} W_e^{\text{loop}}$ $(-)^{\#\text{ of cross}}$ can be -1 only when the loops are non-contracible (non-trivial topology).

- This new relation between the two symmetry is called a **mixed anomaly**. **The anomaly prevents the system to have a gap symmetric ground state (*ie* to have a non-degenerate gapped ground state on all closed spaces).**

  In presence of anomaly:

  Gapped ground state must spontaneously break some symmetry

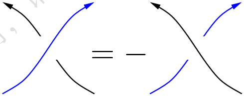
  The symmetric state must be gapless

Anomaly through patch symmetry operator

- A p-symmetry is generated by operators on co-dimension-p closed manifolds. A **patch symmetry operator** is given by part of co-dimension-p closed manifolds – co-dimension-p open manifolds.

- **Example**: In 2-dim space, a 1-symmetry is generated by closed string operators. Its patch symmetry operators are open string operators.

- For the two 1-symmetries $\mathbb{Z}_2^{(1)e}$ and $\mathbb{Z}_2^{(1)m}$ in toric model, their patch symmetry operators are given by open-string operators $W_e^{\text{open-string}}$, $W_m^{\text{open-string}}$



- The boundaries of the patch symmetry operators correspond to the e-particle and the m-particle in the toric model.

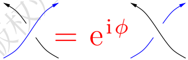
- e-particle and the m-particle have a non-trivial mutual statistics π between them: $W_e^{\text{open-string}} W_m^{\text{open-string}} = -W_m^{\text{open-string}} W_e^{\text{open-string}}$ if strings cross

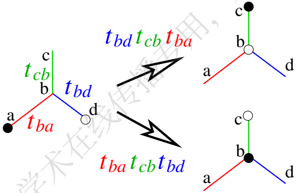
  Wen arXiv:1812.02517

  $\to$ mixed anomaly between two 1-symmetries $\mathbb{Z}_2^{(1)e}$ and $\mathbb{Z}_2^{(1)m}$.

Types of higher symmetries

- The boundaries of patch symmetry operators correspond to topological excitations. **If those topological excitations have trivial mutual statistics and trivial self statistics, then the symmetries are anomaly-free.**

- The mutual statistics ϕ can be calculated from the algebra of patch symmetry operators 

- The self statistics θ can also be calculated from the algebra of patch symmetry operators 

Types of higher symmetries

- The boundaries of patch symmetry operators correspond to topological excitations. **If those topological excitations have quantum dimension-1 (*ie* are Abelian anyons), then the symmetries are higher-form symmetries described by higher group.**

  Nussinov Ortiz, arXiv:cond-mat/0702377 (called gauge-like symmetry)

  Gaiotto Kapustin Seiberg Willett, arXiv:1412.5148 (called higher-form symmetry)

- The boundaries of patch symmetry operators correspond to topological excitations. **If those topological excitations have quantum dimension larger then 1 (*ie* are non-Abelian anyons), then the symmetries are algebraic higher symmetries (also called non-invertible higher symmetries) beyond higher group.** *What mathematical theory describes algebraic higher symmetries?*

  Ji Wen arXiv:1912.13492; Kong Lan Wen Zhang Zheng arXiv:2005.14178

  Fröhlich Fuchs Runkel Schweigert arXiv:hep-th/0607247; Chang Lin Shao Wang Yin arXiv:1802.04445; Thorngren Wang arXiv:1912.02817 (for 1+1D algebraic 0-symmetries)

Non-Abelian anyon and non-Abelian topolocal order

- Topological excitations has internal degrees of freedom, characterized by **quantum dimension** d.

- $d = 1 \to$ Abelian anyon
- $d > 1 \to$ Non-Abelian anyon (or $d \notin \mathbb{Z}$)

Example: spin-$\frac{1}{2}$ has 2 degrees of freedom (the spin quantum states form a 2-dim vector space). Quantum dimension for spin-$\frac{1}{2}$ is $d = 2$.

- The quantum dimension d for a non-Abelian anyon may not be an integer.

Example: A particle carrying a **Majorana zero mode** is a non-Abelian anyon – Ising anyon. It has $\sqrt{2}$ degrees of freedom (the internal states of n Ising anyons form a $\frac{1}{2}(\sqrt{2})^n$-dim vector space). Quantum dimension for Ising anyon is $d = \sqrt{2}$.

2+1D Abelian topological order

- Laughlin filling fraction $\nu = 1/m$ state of electrons:

$$\Psi_{\text{grnd}} = \prod_{i<j}(z_i - z_j)^m \mathrm{e}^{-\frac{1}{4}\sum |z_i|^2}, \quad z_i = x_i + \mathrm{i}\, y_i$$

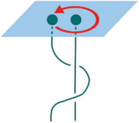
- A charge-1 electron (a hole) excitation at ξ:

$$\Psi_\xi = \prod_i (\xi - z_i)^m \prod_{i<j}(z_i - z_j)^m \mathrm{e}^{-\frac{1}{4}\sum |z_i|^2}$$

- A charge-$\frac{1}{m}$ quasiparticle excitation at ξ, which also carry 2π-flux:

$$\Psi_\xi = \prod_i (\xi - z_i) \prod_{i<j}(z_i - z_j)^m \mathrm{e}^{-\frac{1}{4}\sum |z_i|^2}$$

- As a charge-flux bound state, it carries a fraction statistics



$$\theta = \text{charge} \times \text{flux} \times \frac{1}{2}\text{turn} = \frac{1}{m} \times 2\pi \times \frac{1}{2} = \frac{\pi}{m}.$$

Most general 2+1D Abelian topological order

- K-matrix wave function of κ-layer 2d electrons ($\kappa = \dim(K)$):

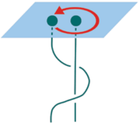
$$\Psi_K = \prod_{i<j, I=1,\cdots,\kappa} (z_i^I - z_j^I)^{K_{II}} \prod_{i,j,I<J} (z_i^I - z_j^J)^{K_{IJ}} \mathrm{e}^{-\frac{1}{4}\sum |z_i^I|^2}$$

where K is a symmetric integer matrix, with even diagonals (assuming electrons are bosons). Blok Wen Phys. Rev. B, **42** 8145 (90)

- An excitation at ξ is labeled by an integer vector $\boldsymbol{l}$:

$$\Psi_\xi = \prod_{i, I=1,\cdots,\kappa} (\xi - z_i^I)^{\boldsymbol{l}_I} \prod_{i<j,I} (z_i^I - z_j^I)^{K_{II}} \prod_{i,j,I<J} (z_i^I - z_j^J)^{K_{IJ}} \mathrm{e}^{-\frac{1}{4}\sum |z_i^I|^2}$$

- Two quasi-particles differ by a bosonic electron are equivalent: $\boldsymbol{l} \sim \boldsymbol{l} + \boldsymbol{K}_{I\text{-col}}$ (column vectors of K). Number of inequivalent quasi-particles $= |\det(K)|$



- Self and mutual statistics

$$\theta_{\boldsymbol{l}} = \pi \boldsymbol{l}^\top K^{-1} \boldsymbol{l}, \quad \phi_{\boldsymbol{l},\boldsymbol{l}'} = 2\pi \boldsymbol{l}^\top K^{-1} \boldsymbol{l}'.$$

- κ edge modes. Signs of K-eigenvalues $\to$ left/right moving modes.

An 8-layer quantum Hall state with no anyons

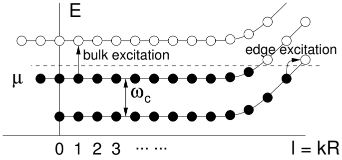
- An even-diagonal symmetric K-matrix with $\det(K) = 1$

$$K_{E_8} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

It gives rise to an E_8 **topological order** which has no anyons (no topological excitations).

Such a topological order is non-trivial since its has 8 right moving edge modes (more precisely, # of right modes − # of left modes = 8). So the E_8 topological order is almost trivial.

IQH state with 2 filled Landau level described by $K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has 2 right moving edge modes.



First examples of non-Abelian topo. order (1991)

Let $\chi_m(\{z_i\})$ be the many-body wavefunction of m filled Landau levels, where $z_i = x_i + \mathrm{i}\, y_i$. $\chi_1 = \prod(z_i - z_j)\mathrm{e}^{-\frac{1}{4}\sum |z_i|^2}, \ \nu = 1$

- $SU(m)_k$ state via slave-particle — Wen PRL **66** 802 (1991)

$$\Psi_{SU(3)_2} = (\chi_2)^3, \ \nu = 2/3; \qquad \Psi_{SU(2)_2} = \chi_1(\chi_2)^2, \ \nu = 1/2;$$

$\to$ Effective $SU(2)_2$, $SU(3)_2$ Chern-Simons theory
$\to$ non-abelian statistics

- Pfaffien state via CFT correlation — Moore-Read NPB **360** 362 (1991)

$$\Psi_{\text{Pfa}} = \mathcal{A}[\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots]\prod(z_i - z_j)^2\mathrm{e}^{-\frac{1}{4}\sum |z_i|^2}, \qquad \nu = 1/2$$

$\to$ Conformal block $\to$ non-abelian statistics.

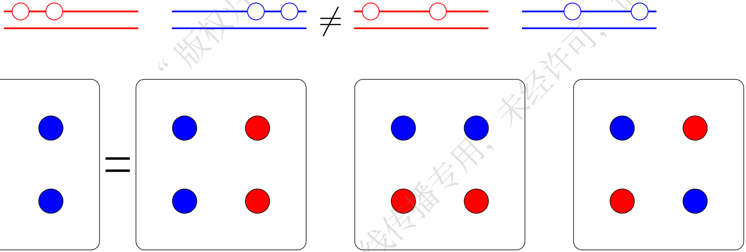
- The $\Psi_{SU(2)_2}$ and Ψ_{Pfa} states have **Ising anyons**, with $d = \sqrt{2}$
- The $\Psi_{SU(3)_2}$ state has **Fibonacci anyons**, with $d = \frac{\sqrt{5}+1}{2}$

Non-Abelian statistics in $\chi_1\chi_2^2$

Topo. degeneracy even when all the quasiparticles are fixed.

- The ground state $\chi_1(\chi_2)^2 = \chi_1\chi_2\chi_2$ is non-degenerateon S^2.
- Degeneracy D_{deg} of 4 trapped quasiparticles at x_1, x_2, x_3, x_4: many different wave functions:

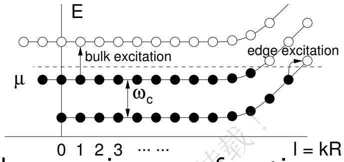
$$\chi_1\chi_2^{x_1x_2}\chi_2^{x_3x_4} \neq \chi_1\chi_2^{x_1x_3}\chi_2^{x_2x_4}$$



- The above represent a topological degeneracy $D_{\text{deg}} = 2$, since locally the two wave functions $\chi_1\chi_2^{x_1}\chi_2$ and $\chi_1\chi_2\chi_2^{x_1}$ are identical.

How to realize non-Abelian FQH states

- IQH = filled Landau levels
- FQH = partially filled Landau levels

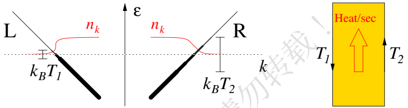


- We can realize (non-)Abelian FQH
  $\Psi_{\nu=1/3} = (\chi_1)^3$ partially filled LL, 3rd order zero in wave function
  $\Psi_{\nu=2/5} = (\chi_1)^2\chi_2$ when the first 2 LLs are degenerate
  $\Psi_{SU(2)_2} = \chi_1(\chi_2)^2$ when the first 3 LLs are degenerate
  $\Psi_{SU(3)_2} = (\chi_2)^3$ when the first 4 LLs are degenerate
- $\Psi_{SU(2)_2} = \chi_1(\chi_2)^2$ contains a neutral fermionic quasiparticle ψ.

  Now we consider $\Psi_{SU(2)_2} = \chi_1(\chi_2)^2$ state with many neutral fermionic quasiparticles ψ, and let the neutral fermionic quasiparticles to form different IQH states, then we can obtain Ψ_{Pfa}, as well as $\Psi_{\text{PH-Pfa}}$, $\Psi_{\overline{\text{Pfa}}}$, $\Psi_{\overline{SU(2)_2}}$, etc.
- $\Psi_{\text{PH-Pfa}}$ is realized in the 2nd LL in experiment.

How to detect non-Abelian FQH states

- The central charge c describes the low energy many-body density of states, *ie* the low temperature specific heat $C = c\frac{\pi}{6}\frac{k_B^2 T}{v\hbar}$
- Thermal Hall conductivity $\kappa = vC = c\frac{\pi}{6}\frac{k_B^2 T}{\hbar}$ is independent of any material properties.

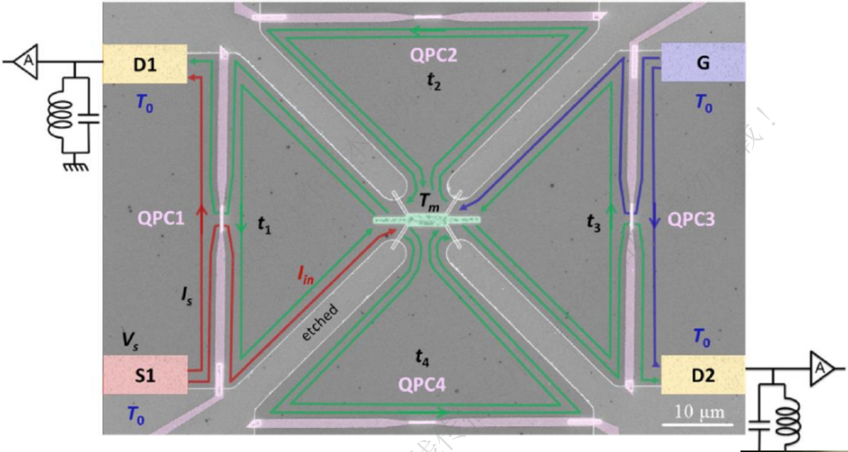


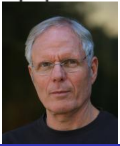
- Several non-Abelian states:

$\Psi^{\text{IQH}}_{\nu=n} = \chi_n$	$c = n$	Abelian
$\Psi^{\text{FQH}}_{(331)}$	$c = 2$	Abelian
$\chi_1(\chi_2)^2$	$c = \frac{5}{2}$	non-Abelian
$\mathcal{A}[\frac{1}{z_1-z_2}\frac{1}{z_3-z_4}\cdots]\prod(z_i-z_j) = \Psi^{\text{FQH}}_{(p+\text{i}p)}$	$c = \frac{3}{2}$	non-Abelian
$\Psi^{\text{FQH}}_{(p-\text{i}p)}$	$c = \frac{1}{2}$	non-Abelian
$\Psi^{\text{SC}}_{(p+\text{i}p)}$	$c = \frac{1}{2}$	non-Abelian

- All Abelian FQH have central charge $c \in \mathbb{Z}$ Wen-Zee 92

Experimental measurement of central charge





Banerjee, Heiblum, etc arXiv:1710.00492 found
$c = 2.56$ 18mK, $c = 2.64$ 15mK, $c = 2.76$ 12mK,
for the $\nu = 5/2$ state in GaAs-AlGaAs hetero structure.