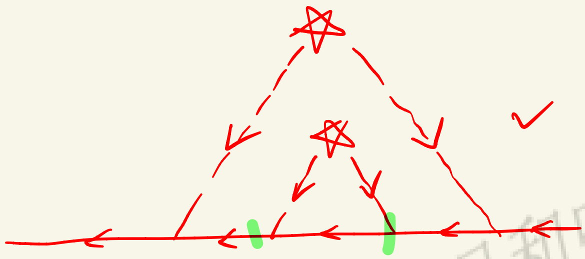
$$\Sigma_k^{SCBA} = \mathrm{Im} \sum_{k'} \frac{|U_{kk'}^{SCBA}|^2}{ik_n - \xi_{k'} - \Sigma_k^{SCBA}} \quad (4.142)$$

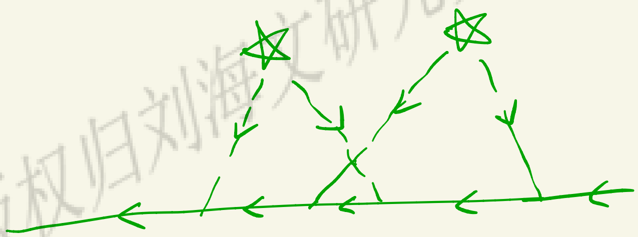
$\ell \gg \lambda$ ← wave length ℓ mean free path.

$k_F \ell \gg 1$ weak disorder



✓

dominant



$k_F \underline{\underline{\ell}} \ll 1$ dominant

$\boxed{\ell \ll \lambda}$ Strong disorder

$$\ell \ll \frac{1}{k_F} \sim \lambda$$

↑ wave length

☆ hydrodynamics ☆

$$\frac{1}{\tau_k} = 2\pi \sum_{k'} |t_{k,k'}|^2 \, \delta(\xi_k - \xi_{k'})$$

$$A_k^{TMA} = \frac{1/\tau_k}{(\omega - \xi_k)^2 + \frac{1}{4\tau_k^2}}$$

Lesson 10

Feynman diagrams for interacting Fermions.

$$H = H_0 + W \qquad H_0 = \sum_{\nu_1 \nu_2} C^\dagger_{\nu_1\sigma} h_{0,\nu_1\nu_2} C_{\nu_2\sigma}$$

$$W = \frac{1}{2} \sum_{\sigma_1 \sigma_2} \int dr_1 dr_2 \, \Psi^\dagger(\sigma_1 r_1) \Psi^\dagger(\sigma_2, r_2) W(\sigma_2 r_2, \sigma_1 r_1) \Psi(\sigma_2 r_2) \Psi(\sigma_1 r_1)$$

$$= \frac{1}{2} \sum_{\nu_1 \nu_2 \nu_3 \nu_4} V_{\nu_1\nu_2,\nu_3\nu_4}\, a^{+}_{\nu_1\sigma_1} a^{+}_{\nu_2\sigma_2} a_{\nu_4\sigma_2} a_{\nu_3\sigma_1} \qquad (4.143)$$

$$\mathcal{G}(\underbrace{\sigma_b, \nu_b, \tau_b}_{b}; \underbrace{\sigma_a, \nu_a, \tau_a}_{a}) \equiv -\langle T_\tau\, \Psi(\sigma_b \nu_b \tau_b)\, \Psi^{+}(\sigma_a \nu_a \tau_a)\rangle$$

Interaction picture.

$$\langle T_\tau\, A(\tau) B(\tau')\rangle = \frac{1}{Z} \mathrm{Tr}\left[e^{-\beta H_0}\, T_\tau\left(\hat{U}(\beta,0)\, \hat{A}(\tau)\, \hat{B}(\tau')\right)\right]$$

$$\mathcal{G}(b;a) = -\frac{\langle T_\tau\left(\hat{U}(\beta,0)\hat{A}(b)\hat{B}(a)\right)\rangle_0}{\langle \hat{U}(\beta,0)\rangle_0} \qquad (4.144)$$

$$= -\frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^{\infty} d\tau_1 \cdots \int_0^{\infty} d\tau_n \langle T_\tau[\hat{W}(\tau_1)\cdots\hat{W}(\tau_n)\, \Psi(b)\, \Psi^{+}(a)\rangle_0}{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^{\beta} d\tau_1 \cdots \int_0^{\beta} d\tau_n \langle T_\tau[\hat{W}(\tau_1)\cdots\hat{W}(\tau_n)]\rangle_0} \qquad (4.145)$$

2n+1 - particle GF

$$= -\frac{\sum_{n=0}^{\infty} \frac{(-1/2)^n}{n!} \int d1\, d1' \ldots dn\, dn'\, W_{11'} \cdots W_{nn'} \langle T_\tau[\Psi_1^{+}\Psi_{1'}^{+}\Psi_{1'}\Psi_1 \cdots \Psi_n^{+}\Psi_{n'}^{+}\Psi_{n'}\Psi_n\, \Psi_b\, \Psi_a^{+}]\rangle_0}{\sum_{n=0}^{\infty} \frac{(-1/2)^n}{n!} \int d1\, d1' \ldots dn\, dn'\, W_{11'} \cdots W_{nn'} \langle T_\tau[\Psi_1^{+}\Psi_{1'}^{+}\Psi_{1'}\Psi_1 \cdots \Psi_n^{+}\Psi_{n'}^{+}\Psi_{n'}\Psi_n]\rangle_0} \qquad (4.146)$$

2n - particle GF.

---

non-interacting H_0

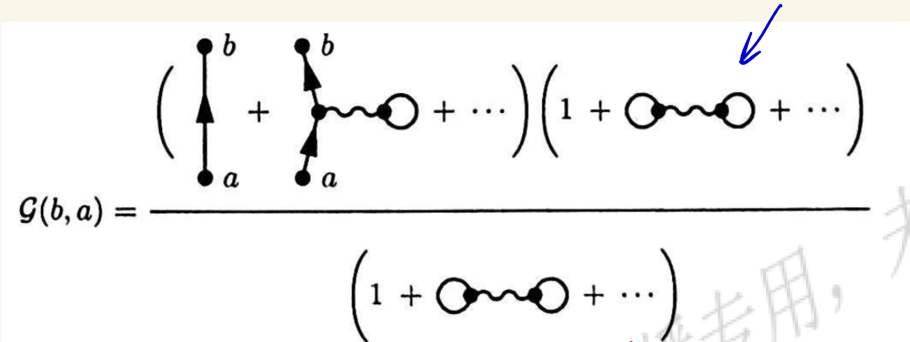
$$\mathcal{G}_0^{(n)}(\nu_1\tau_1, \cdots, \nu_n\tau_n; \nu_1'\tau_1', \cdots, \nu_n'\tau_n')$$

$$= (-1)^n \langle T_\tau[\hat{C}_{\nu}(\tau_1)\cdots\hat{C}_{\nu_n}(\tau_n)\, C^{+}_{\nu_n'}(\tau_n')\cdots C^{+}_{\nu_1'}(\tau_1')]\rangle_0$$

$$\mathcal{G}_0^{(n)}(1,\cdots,n; 1',\cdots,n') = \begin{vmatrix} \mathcal{G}_0(1,1') & \cdots & \mathcal{G}_0(1,n') \\ \vdots & & \vdots \\ \mathcal{G}_0(n,1') & \cdots & \mathcal{G}_0(n,n') \end{vmatrix}_{B,F}$$

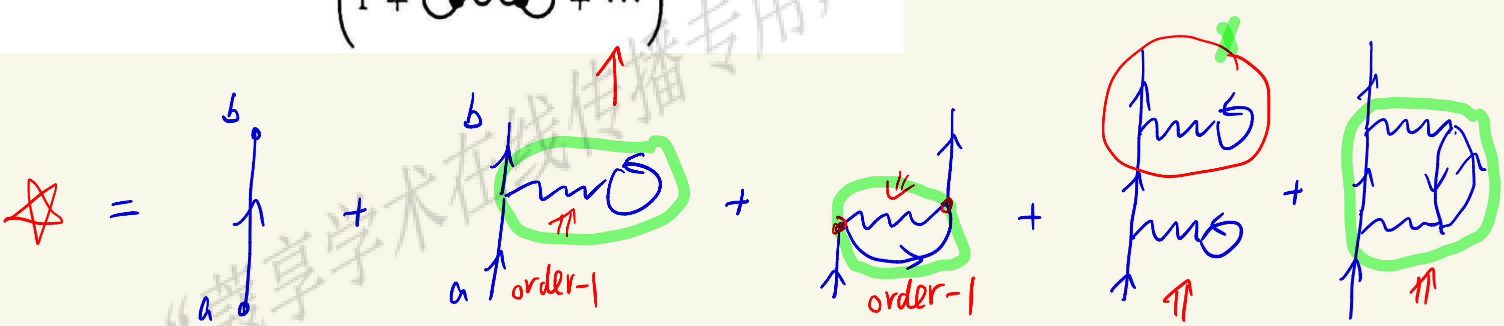
From Wick's theorem,

$$\mathcal{G}(b,a) = \sum_{n=0}^{\infty} \frac{(-1/2)^n}{n!} \int d1\, d1' \cdots dn\, dn' \; W_{11'} \cdots W_{nn'} \times \frac{\overbrace{\begin{vmatrix} g^0(b,a) & g^0(b,1) & g^0(b,1') & \cdots & g^0(b,n') \\ g^0(1,a) & g^0(1,1) & g^0(1,1') & \cdots & g^0(1,n') \\ g^0(1',a) & g^0(1',1) & g^0(1',1') & \cdots & g^0(1',n') \\ \vdots & \vdots & & & \\ g^0(n',a) & g^0(n',1) & g^0(n',1') & \cdots & g^0(n',n') \end{vmatrix}}^{(2n+1)\times(2n+1)}}{\sum_{n=0}^{\infty} \frac{(-1/2)^n}{n!} \int d1\, d1' \cdots dn\, dn' \; W_{11'} \cdots W_{nn'} \times \underbrace{\begin{vmatrix} g^0(1,1) & g^0(1,1') & \cdots & g^0(1,n') \\ g^0(1',1) & g^0(1',1') & \cdots & g^0(1',n') \\ \vdots & & & \\ g^0(n',1) & g^0(n',1') & \cdots & g^0(n',n') \end{vmatrix}}_{2n\times 2n}} \qquad (4.147)$$

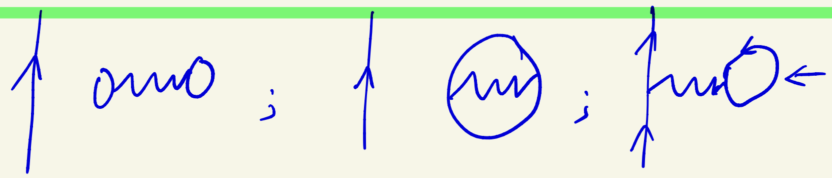


(4.148)

reducible Irreducible

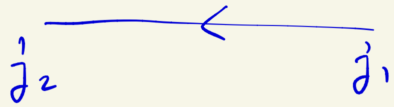


nominator:

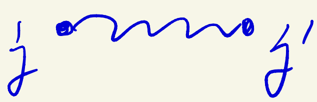


disconnected diagrams; Connected

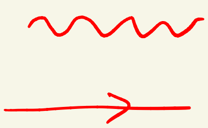
* Feynmann Rules *

①  $= g^0(j_2, j_1)$

$j_1 \equiv (\sigma_{j_1}, r_{j_1}, \tau_{j_1})$

②  $\equiv -W_{jj'}$

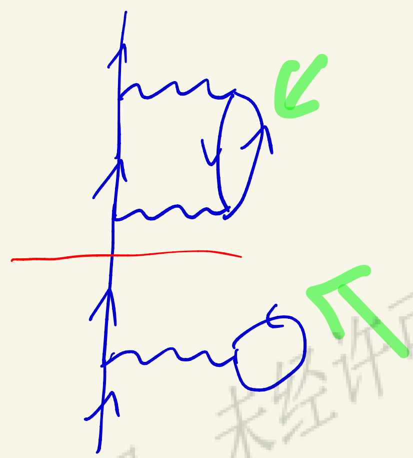
③ $(-1)^F$; F number of fermion loops.

④ n order diagrams: n interaction lines 

$2n+1$ fermion lines

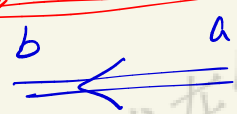
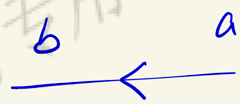
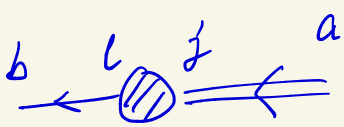
---

irreducible 

reducible diagram 

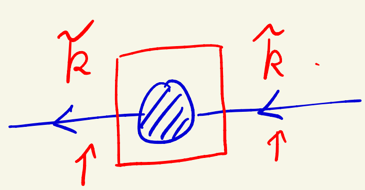
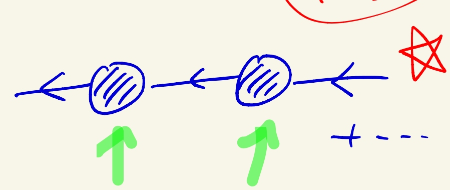
(4.149)

Dyson's equation:

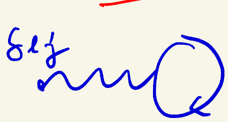
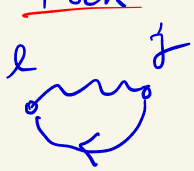
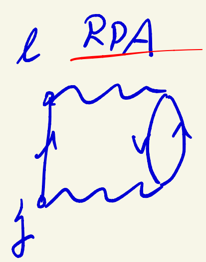
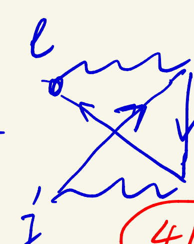
$g(b,a) \equiv$  $=$  $+$ 

$= g^0(b,a) + \sum_{l,j} g^0(b,l) \Sigma(l,j) g(j,a)$

(4.150)

$=$  $+$  $+$  $+ \cdots$

Self-energy: Hartree Fock RPA

$\Sigma(l,j) \equiv$  $+$  $+$  $+$  $+\cdots$

$\equiv$ 

(4.151)

Feynman Rules in Momentum-frequency space.

$i\tilde{k}\tilde{r} = i k\cdot r - i k_n \tau$

$\tilde{r} \equiv (r, \tau)$

$\tilde{k} \equiv (k, i k_n)$

$$\int d\tilde{r}\; g^0_\sigma(\tilde{r}_2;\tilde{r})\, g^0_\sigma(\tilde{r};\tilde{r}_1)\, W(\tilde{r}_3;\tilde{r})$$

(4.152)

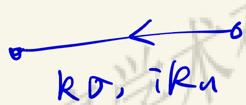
$$\int d\tilde{r}\; g^0_\sigma(\tilde{r}_2;\tilde{r})\, g^0_\sigma(\tilde{r};\tilde{r}_1)\, W(\tilde{r}_3;\tilde{r})$$

$$= \frac{1}{(\beta V)^3}\sum_{\tilde{k}\tilde{p}\tilde{q}} g^0_\sigma(\tilde{p})\, g^0_\sigma(\tilde{k})\, \tilde{W}(\tilde{q}) \int d\tilde{r}\; e^{i[\tilde{p}(\tilde{r}_2-\tilde{r})+\tilde{k}(\tilde{r}-\tilde{r}_1)+\tilde{q}(\tilde{r}_3-\tilde{r})]}$$

$$= \frac{1}{(\beta V)^2}\sum_{\tilde{k}\tilde{q}} g^0_\sigma(\tilde{k}-\tilde{q})\, g^0_\sigma(\tilde{k})\, W(\tilde{q})\; e^{i[\tilde{k}(\tilde{r}_2-\tilde{r}_1)+\tilde{q}(\tilde{r}_3-\tilde{r}_2)]} \qquad (4.153)$$

$\tilde{p} = \tilde{k} - \tilde{q}$

Feynman Rules in k-ω space.

① 

$\equiv g^0(\vec{k}, ik_n)$

② 

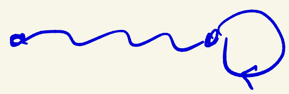
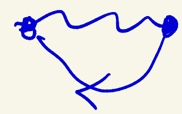
$\equiv -W(q)$

③ vertex → four momentum conserve.

④ order-n → $2n+1$ fermions lines

n interaction lines.

⑤ fermion loops number F; $(-1)^F$

⑥  ;  " same-time " $e^{ik_n\eta}$.

⑦ Internal momentum $\hat{p}$; sum $\sum_{\hat{p}\sigma'}$; (4.154)

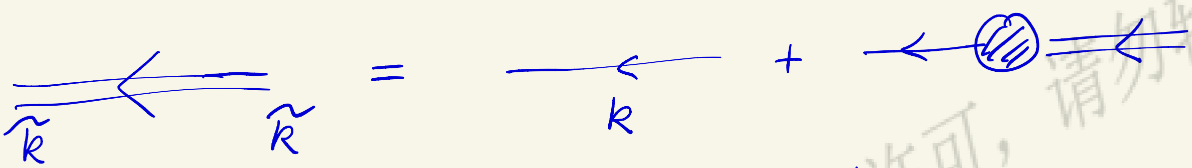
multiply the volume factor $\frac{1}{\beta V}$

* Fetter-Walecka's Book

due to the four-momentum conservation;

$$\Sigma_\sigma(\hat{k},\hat{k}') = \delta_{\hat{k}\hat{k}'}\Sigma_\sigma(\hat{k})$$

$$\mathcal{G}_\sigma(\hat{k}) = \mathcal{G}^0_\sigma(\hat{k}) + \mathcal{G}^0_\sigma(\hat{k})\,\Sigma_\sigma(\hat{k})\,\mathcal{G}_\sigma(\hat{k})$$



(4.155)

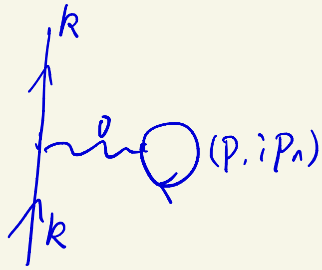
$$\mathcal{G}_\sigma(k,ik_n) = \frac{\mathcal{G}^0_\sigma(k,ik_n)}{1-\mathcal{G}^0_\sigma(k,ik_n)\,\Sigma_\sigma(k,ik_n)}$$

$$= \frac{1}{\mathcal{G}_0^{-1} - \Sigma_\sigma(k,ik_n)}$$

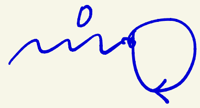
$$= \frac{1}{ik_n - \xi_k - \underline{\Sigma_\sigma(k;ik_n)}}$$

Examples of Feynman diagrams.

① Hartree self-energy diagram

$$g^H_\sigma(k;ik_n) =$$ 

$$\cdot e^{ip_n\eta}$$

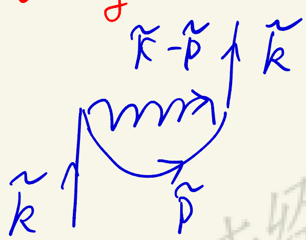
$$\Sigma^H_\sigma(k;ik_n) \equiv$$  $$= \frac{-1}{\beta V}\sum_{\sigma'}\sum_{\vec p}\sum_{ip_n}[-W(0)]\, g^0_{\sigma'}(\vec p, ip_n)$$

$$= \frac{2W(0)}{\beta}\int\frac{d\vec p}{(2\pi)^3}\sum_{ip_n}\frac{e^{ip_n\eta}}{ip_n - \xi_p} \qquad \text{ETC}$$

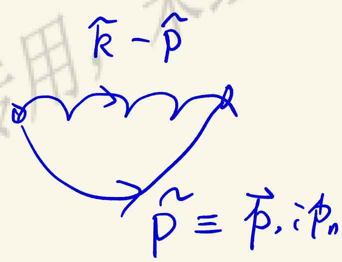
$$= 2W(0)\int\frac{d\vec p}{(2\pi)^3}\, n_F(\xi_p) = W(0)\cdot\frac{N}{V}$$

Cancel out with the positive background charge.

② Fock self-energy diagram:

$$g^F_\sigma(k;ik_n) =$$ 

(4.157)

$$\Sigma^F_\sigma(k;ik_n) =$$ 

$$= \frac{1}{\beta V}\sum_{\sigma'}\sum_{\vec p}\sum_{ip_n}[-W(\vec k - \vec p)]\,\delta_{\sigma\sigma'}\, g^0_{\sigma'}(p;ip_n)\, e^{ip_n\eta}$$

$$= -\int\frac{d\vec p}{(2\pi)^3} W(\vec k - \vec p)\, n_F(\xi_p)$$

Mean-field derivation of Hartree-Fock:

$$V = \frac{1}{2V}\sum_{\sigma_1\sigma_2}\sum_{k_1k_2q} V_q\, a^+_{k_1+q\,\sigma_1} a^+_{k_2-q\,\sigma_2} a_{k_2\sigma_2} a_{k_1\sigma_1}$$

$$\langle c_\nu^+ c_\mu^+ c_{\mu'} c_{\nu'} \rangle_0 = \langle c_\nu^+ c_{\nu'} \rangle_0 \langle c_\mu^+ c_{\mu'} \rangle_0 - \langle c_\nu^+ c_{\mu'} \rangle_0 \langle c_\mu^+ c_{\nu'} \rangle_0$$

$$<$$

$$\begin{aligned} c_\nu^+ c_\mu^+ c_{\mu'} c_{\nu'} \simeq\ & c_\nu^+ c_{\nu'} \cdot \langle c_\mu^+ c_{\mu'} \rangle_{MF} + \langle c_\nu^+ c_{\nu'} \rangle \cdot c_\mu^+ c_{\mu'} \\ & - \langle c_\nu^+ c_{\nu'} \rangle_{MF} \langle c_\mu^+ c_{\mu'} \rangle_{MF} \qquad H \\ & - c_\nu^+ c_{\mu'} \langle c_\mu^+ c_{\nu'} \rangle_{MF} - \langle c_\nu^+ c_{\mu'} \rangle_{MF} c_\mu^+ c_{\nu'} \\ & + \langle c_\nu^+ c_{\mu'} \rangle_{MF} \langle c_\mu^+ c_{\nu'} \rangle_{MF} \qquad F \end{aligned}$$

thus: Hartree type:

$$V_H = 2 \times \frac{1}{2V} \sum_{\sigma_1 \sigma_2} \sum_{k_1 k_2 q} V_q\, a^+_{k_1+q,\sigma_1} a_{k_1 \sigma_1} \langle a^+_{k_2-q,\sigma_2} a_{k_2 \sigma_2} \rangle_{MF}$$

$$n_{k_2,\sigma_2}\, \delta_{q=0}$$

$$= \frac{1}{V} \sum_{\sigma_1 \sigma_2} \sum_{k_1 k_2} V_0\, a^+_{k_1 \sigma_1} a_{k_1 \sigma_1} \cdot \bar{n}_{k_2 \sigma_2} \qquad (4.159)$$

Fock type

$$q = k_2 - k_1$$

$$V_F = 2 \times \frac{-1}{2V} \sum_{\sigma_1 \sigma_2} \sum_{k_1 k_2 q} V_q\, a^+_{k_1+q,\sigma_1} a_{k_2 \sigma_2} \langle a^+_{k_2-q,\sigma_2} a_{k_1 \sigma_1} \rangle_{MF}$$

$$k_2 - q = k_1$$

$$\sigma_1 = \sigma_2$$

$$= \frac{-1}{V} \sum_{\sigma_1 \sigma_2} \sum_{k_1 k_2} V(k_2 - k_1)\, \hat{n}_{k_2 \sigma_1} \cdot \delta_{\sigma_1 \sigma_2}\ \bar{n}_{k_1 \sigma_1}$$

$$\mathcal{H}_{HF} = \sum_{k\sigma} \left\{ \epsilon_k + \sum_{k'\sigma'} \left[V(0) - \delta_{\sigma\sigma'} V(k-k') \right] \bar{n}_{k'\sigma'} \right\} \cdot c^+_{k\sigma} c_{k\sigma}$$

$$V_{HF} = \sum_{k'\sigma'} \left(V(0) - \delta_{\sigma\sigma'}\right) V(k-k')\, \bar{n}_{k'\sigma'}$$

$$= -\frac{e^2 k_F}{4\pi^2 \varepsilon_0}\left[1 + \frac{k_F^2 - k^2}{2k_F k} \ln\left|\frac{k+k_F}{k-k_F}\right|\right]$$

$\boxed{\bar{n}_{k'\sigma'} = \theta(k_F - k')}$

4.160

$$\epsilon_{HF} = \xi_k + \Sigma(k) \qquad \Sigma(k) = V_{HF}$$

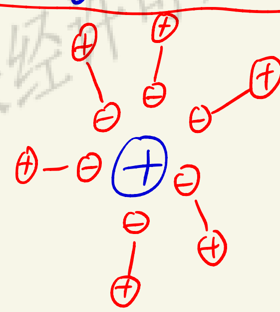
$$\Sigma(k=k_F) = \frac{-e^2 k_F}{4\pi^2 \varepsilon_0}$$

4.161

$$\left.\frac{\partial \epsilon_{HF}}{\partial k}\right|_{k=k_F} = \underbrace{\left.\frac{\partial \xi_k}{\partial k}\right|_{k=k_F}}_{\propto k_F \ \uparrow \text{ finith.}} + \underbrace{\left.\frac{\partial \Sigma}{\partial k}\right|_{k=k_F}}_{\text{divergent.}} \to \infty$$

$\frac{\partial \epsilon_{HF}}{\partial k}$ divergent; fermi energy DOS dos is zero.

* Coulomb screening $\Longleftarrow$



$\frac{1}{r} \Rightarrow \frac{1}{r} e^{-r/\lambda}$

electron-hole excitation.

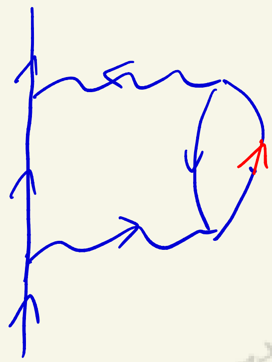
* Density functional theory: microscopic theory.

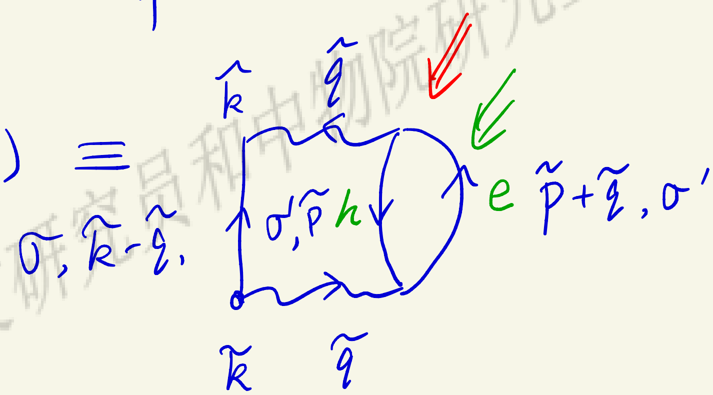
* Landau's Fermi-Liquid theory. phenomenological theory.

Pair-bubble diagram; Bohm-Pines; 1950-1953

EOM

Gell-mann; Brueckner 1957; Feynman Diagram

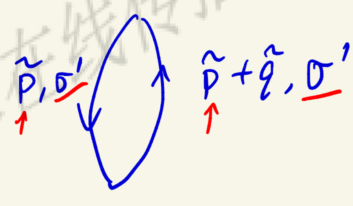
$g^P_\sigma(k; ik_n) \equiv$  (4.162)

$\Sigma^P_\sigma(k; ik_n) \equiv$ 

$$= \frac{(-)1}{(\beta V)^2}\sum_{\sigma' p q}\sum_{ip_n q_n}[-W(q)]^2\, g^0_{\sigma'}(p, ip_n)\, g^0_{\sigma'}(p+q, ip_n+iq_n)$$

$$\cdot\, g^0_\sigma(k-q, ik_n-iq_n) \quad (4.163)$$

$$= \frac{1}{\beta}\sum_{iq_n}\int\frac{dq}{(2\pi)^3}[W(q)]^2 \cdot \pi^0(q, iq_n)\, g^0_\sigma(k-q, ik_n-iq_n)$$

$\pi^0(q, iq_n) \equiv$  (4.164)

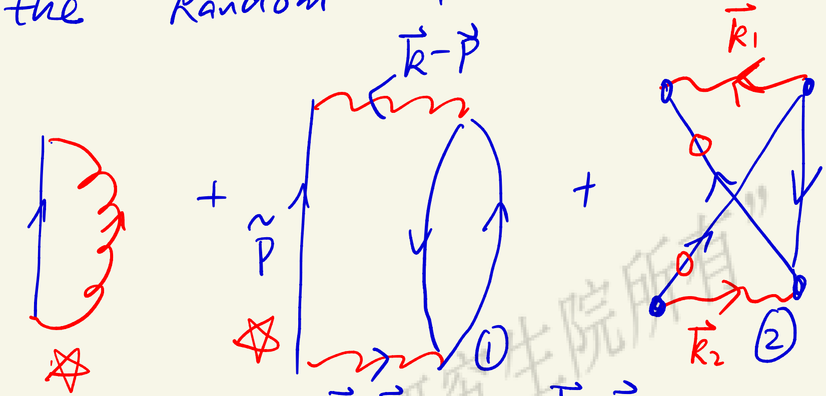
χ^0

$$= \frac{-2}{\beta}\sum_{ip_n}\int\frac{d\tilde{p}}{(2\pi)^3}\frac{1}{ip_n+iq_n-\xi_{p+q}}\cdot\frac{1}{ip_n-\xi_p}$$

Interacting-electron system: $= \dfrac{e^2}{4\pi\varepsilon_0|r|}e^{-q_s\cdot|r|}$

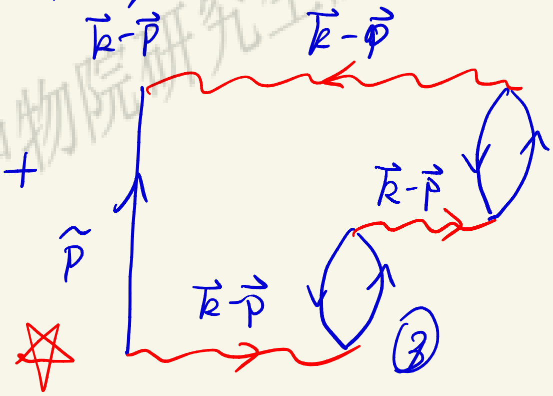
Yukawa-potential $W(\vec{r}) = \dfrac{e^2}{4\pi\varepsilon_0|r|}e^{-\frac{|r|}{\lambda}}$ (4.165) discussion on the Random - phase - Approximation.

$\Sigma_\sigma(\underline{\vec{k}}, ik_n) =$



4.166

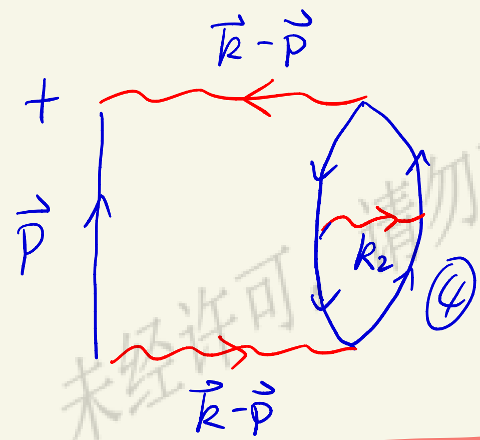
Cancel - out with positive background.



4.167

① $|W(\vec{k}-\vec{q})|^2$

divergence $|\vec{k}-\vec{q}| = 0$



② $W(k_1)\,W(k_2)$

divergence : $k_1 = k_2 = 0$

diagram ① larger than diagram ② $\boxed{\text{Bare } W(k) \propto \frac{1}{k^2}}$

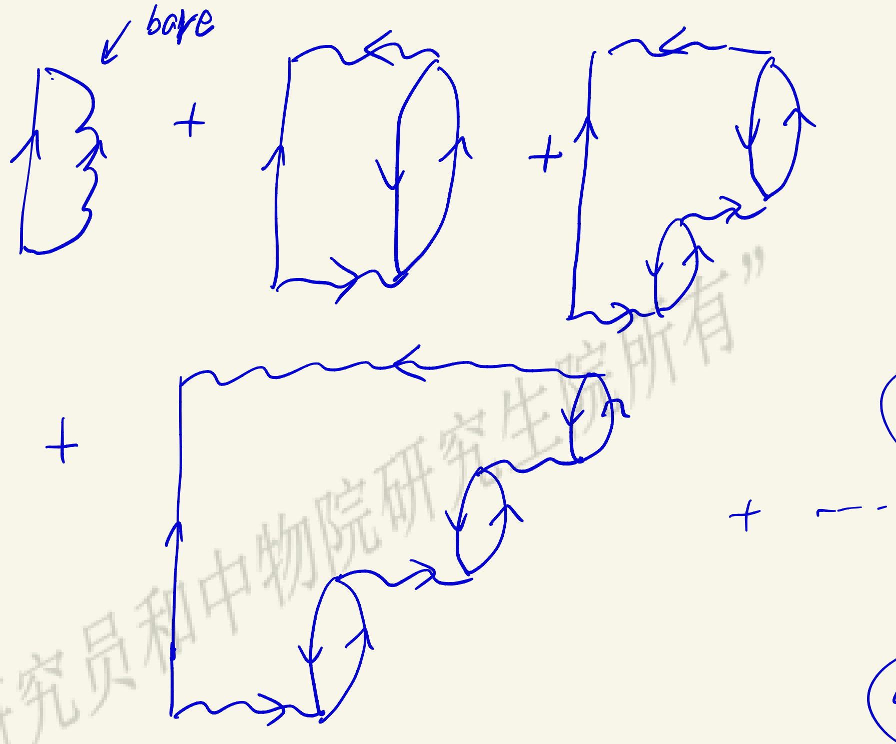
③ $|W(\vec{k}-\vec{p})|^3$; divergence $k-p = 0$;

4.168

④ $|W(k-\vec{p})|^2\, W(k_2)$ divergence $k-p=0$; $k_2 = 0$;

RPA :

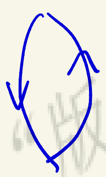
$\Sigma_\sigma^{RPA}(\tilde{k}) \equiv$



(4.169)

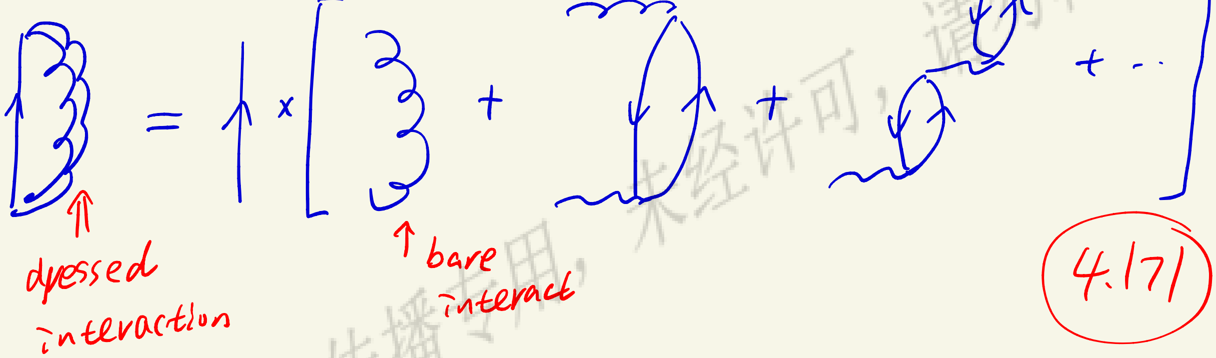
(4.170)

$\chi_0 = -\Pi_0$

 $\equiv -\chi_0(q, iq_n)$

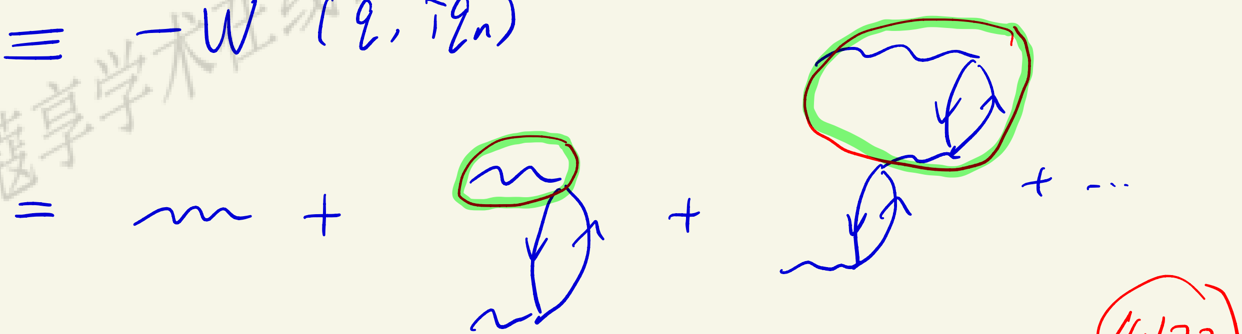
$$\chi_0(q, iq_n) = \frac{2}{\beta}\sum_{ip_n}\int\frac{d\vec{p}}{(2\pi)^3}\,\frac{1}{ip_n + iq_n - \xi_{p+q}}\cdot\frac{1}{ip_n - \xi_p}$$

$\Sigma_\sigma^{RPA} =$

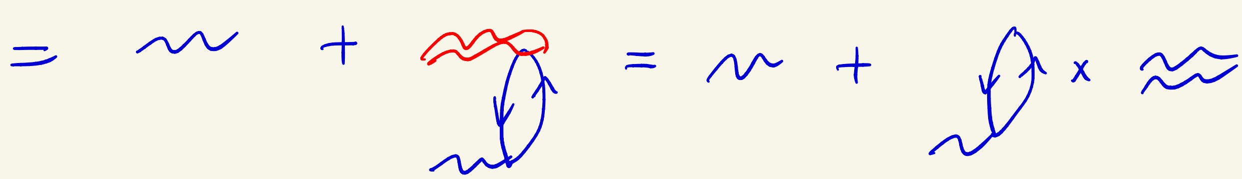


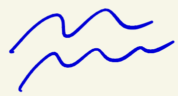
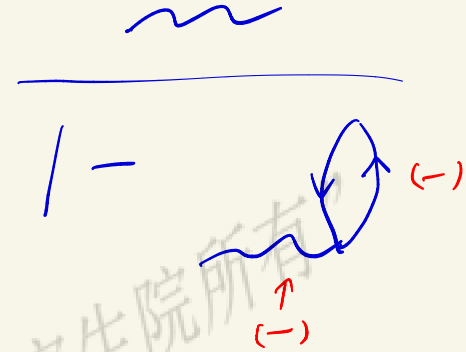
(4.171)

 $\equiv -W^{RPA}(q, iq_n)$



(4.172)

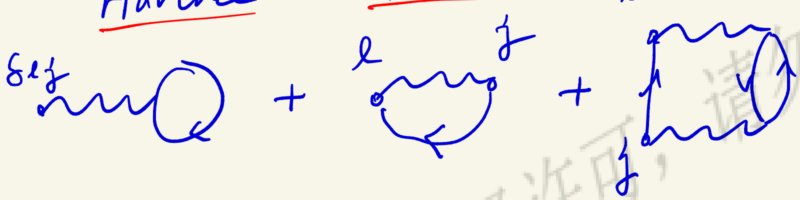


$$-W^{RPA}(q,iq_n) = $$  $$=$$ 

$$= \frac{-W(q)}{1 - W(q)\,\chi_0(q,iq_n)} \qquad (4.173)$$

$$W^{RPA} = \frac{W(q)}{\varepsilon(q,iq_n)} \Rightarrow \varepsilon(q,iq_n) = 1 - W(q)\cdot\chi_0(q,iq_n)$$

* Hartree ✓ Fock ✓ & RPA ☆

$$\Sigma(l,j) \equiv$$ 

Feynman - Rules for above three diagrams.

Lesson 11

Chapter 5 Interacting Electron System.

From the RPA calculation, one can obtain

$$e_0^2 = \frac{e^2}{4\pi\varepsilon_0}$$

$$W^{RPA}(q,iq_n) = \frac{4\pi e_0^2}{q^2 - 4\pi e_0^2\,\chi_0(q,iq_n)} = \frac{\frac{4\pi e_0^2}{q^2}}{\varepsilon(q,iq_n)} \qquad (5.1)$$

Yukawa potential: $V(r) = \frac{e_0^2}{r} e^{-k_s\cdot r}$ (5.2)

$\Downarrow$

ETC: Fourier Transform: $V(q) = \dfrac{4\pi e_0^2}{q^2 + k_s^2}$