Similarly, in Lehmann Spectral representation. (4.60)

$$S(\vec{q},\omega) = \int dt\, e^{i\omega t}\, S(\vec{q},t) = \int dt\, e^{i\omega t} \sum_n |\langle n|\hat{\rho}^+(q)|\phi_0\rangle|^2 \cdot e^{i(E_n-E_0)t}$$

$$= 2\pi \sum_n \langle n|\hat{\rho}^+(q)|\phi_0\rangle\, \delta[\omega-(E_n-E_0)]$$

Zero temprature Fluctuation-Dissipation Theorem. (4.61)

$$\boxed{\chi''(\vec{q},\omega) = -\tfrac{1}{2}[S(\vec{q},\omega) - S(-\vec{q},-\omega)]}$$

response function

dynamical structure factor.

Lesson 8

Equation of Motion EOM

generate a couple of differential equation

generally not closed, → truncate the equation.

Single particle Green's function

$$G^R(rt,r't') = -i\,\theta(t-t')\,\langle[\psi(r,t),\psi^+(r't')]_{B,F}\rangle$$

$$i\partial_t \psi(r,t) = -[H,\psi(r)](t) = \underset{-\frac{1}{2m}\nabla_r^2\psi(r)}{\underline{-[H_0,\psi(r)](t)}} - [V_{int},\psi(r)](t) \quad ②$$

$$i\partial_t G^R(tr,t'r') = -i[i\partial_t\theta(t-t')]\langle[\psi(r,t),\psi^+(r't')]_{B,F}\rangle \quad ①$$

$$+ -i\theta(t-t')\langle[\underline{i\partial_t\psi(r,t)},\psi^+(r',t')]_{B,F}\rangle$$

$$\left(i\partial_t + \frac{1}{2m}\nabla_r^2\right) G^R(tr, t'r') = \underset{①}{\delta(t-t')\delta(r-r')} + \underset{②}{D^R(rt, r't')}$$

$$D^R(rt, r't') = -i\theta(t-t')\left\langle \left[-[V_{int}, \Psi(r)](t), \Psi^+(r't')\right]_{B,F} \right\rangle \quad \textcircled{4.62}$$

If $\underline{H_0 = \sum_{\nu\nu'} t_{\nu\nu'} a^+_{\nu'} a_\nu}$ $\qquad H = H_0 + V_{int}$

$$G^R(\nu t, \nu' t') = -i\theta(t-t')\left\langle [a_\nu(t), a_{\nu'}(t')]_{B,F} \right\rangle$$

$$-[H_0, a_\nu] = \sum_{\nu''} t_{\nu\nu''} a_{\nu''}$$

$$\sum_{\nu''}\left(i\delta_{\nu\nu''}\partial_t - t_{\nu\nu''}\right) G^R(\nu'' t, \nu' t') = \delta(t-t')\delta_{\nu\nu'} + D^R(\nu t, \nu' t') \leftarrow \textcircled{4.63}$$

$$\boxed{e^{i(\vec{p}\cdot\vec{r}-\omega t)}}$$

$$D^R(\nu t, \nu' t') = -i\theta(t-t')\left\langle \left[-[V_{int}, a_\nu](t), a^+_{\nu'}(t')\right]_{B,F} \right\rangle$$

Hamiltonian does not depends on time, $t - t'$

$$\sum_{\nu''}\left[\delta_{\nu\nu''}(\omega + i\eta) - t_{\nu\nu''}\right] G^R(\nu'', \nu; \omega) = \delta_{\nu\nu'} \boxed{+ D^R(\nu, \nu'; \omega)}$$

$$D^R(\nu, \nu'; \omega) = -i\int_{-\infty}^{+\infty} dt\, e^{i(\omega+i\eta)(t-t')}\theta(t-t') \left\langle \left[-[V_{int}, a_\nu](t), a^+_{\nu'}(t')\right]_{B,F} \right\rangle$$

↑ 4.64

Non-interacting case

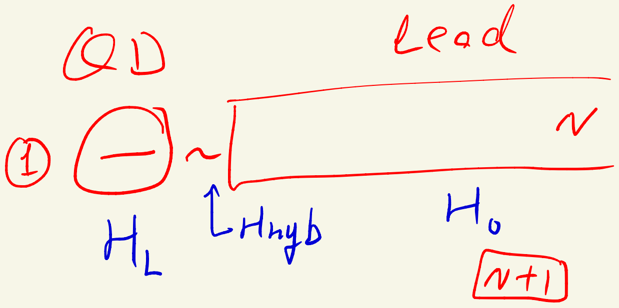
$$\sum_{\nu''}\underbrace{[\delta_{\nu\nu''}(\omega+i\eta)-t_{\nu\nu''}]}\,G_0^R(\nu'',\nu',\omega)=\delta_{\nu\nu'}$$

4.65

$\Rightarrow$ if $t_{\nu\nu'}=\varepsilon_\nu\delta_{\nu\nu'}$

$$G_0^R(\nu,\nu';\omega)\equiv G_0^R(\nu;\omega)\delta_{\nu\nu'}=\frac{1}{\omega-\varepsilon_\nu+i\eta}\delta_{\nu\nu'}$$

Single electron level couple to a continuum



$$H=H_0+H_{hyb}+H_L$$

$$H_0=\sum_\nu \xi_\nu \underline{c_\nu^\dagger c_\nu}$$

$$H_L=\xi_0\,\underline{c_L^\dagger c_L}$$

$$H_{hyb}=\sum_\nu[t_\nu^* c_\nu^\dagger c_L+t_\nu c_L^\dagger c_\nu]$$

two retarded Green's function:

$$G^R(l,l;t-t')=-i\theta(t-t')\langle\{c_l(t),c_l^\dagger(t')\}\rangle$$

$$G^R(\nu,l;t-t')=-i\theta(t-t')\langle\{c_\nu(t),c_l^\dagger(t')\}\rangle$$

$$\left\{\begin{array}{l}(\omega+i\eta-\xi_0)\underline{G^R(l,l,\omega)}-\sum_\nu t_\nu G^R(\nu,l;\omega)=1 \quad \boxed{4.66}\\ (\omega+i\eta-\xi_\nu)G^R(\nu,l;\omega)-t_\nu^* G^R(l,l;\omega)=0\end{array}\right.$$

$$(\omega+i\eta)\vec{\vec{G}}^r-\overset{N+1}{\begin{bmatrix}\xi_0 & t_{01} & \cdots & t_{0N}\\ \vdots & \xi_1 & & \vdots\\ \vdots & & \ddots & \vdots\\ \cdots & & & \xi_N\end{bmatrix}}\vec{\vec{G}}^r=I_{N+1}$$

$$\vec{\vec{G}}^r_{00}=G^R(L,L,\omega)$$

$$\overleftrightarrow{G}^r = \frac{1}{\omega + i\eta - \overleftrightarrow{H}}$$

$$G^R(l,l;\omega) = \frac{1}{\omega - \xi_0 - \Sigma^R(\omega)}$$

$\boxed{468}$

☆ $$\Sigma^R(\omega) = \sum_{\underset{\uparrow}{\nu}} \frac{|t_\nu|^2}{\omega - \xi_\nu + i\eta}$$ self-energy

Anderson model of magnetic impurity

王顺金

运动方程截断方案

↓

量子点 kondo 问题

$$H_c = \sum_{k\sigma} (\varepsilon_k - \mu)\, C^+_{k\sigma} C_{k\sigma}$$

$$H_d + H_U = \sum_{\sigma} (\varepsilon_d - \mu)\, C^+_{d\sigma} C_{d\sigma} + \underline{\boxed{U n_{d\uparrow} n_{d\downarrow}}}$$

$$H_{hyb} = \sum_{k\sigma} \left(t_k \underline{\underline{C^+_{d\sigma} C_{k\sigma}}} + t_k^* \underline{\underline{C^+_{k\sigma} C_{d\sigma}}} \right)$$

$$H = H_c + H_d + H_U + H_{hyb}$$

the EOM for Anderson Model

$$\langle n_{d\sigma} \rangle = \int \frac{d\omega}{2\pi} A(d\sigma, \omega)\, n_F(\omega)$$

$$G^R(d\sigma; t-t') = -i\theta(t-t') \langle \{ C_{d\sigma}(t), C^+_{d\sigma}(t') \} \rangle$$

$$G^R(k\sigma, d\sigma; t-t') = -i\theta(t-t') \langle \{ C_{k\sigma}(t), C^+_{d\sigma}(t') \} \rangle$$

$$\left\{\begin{array}{l} (\omega + i\eta - \varepsilon_d + \mu)\, G^R(d\sigma;\omega) - \sum_k t_k\, G^R(k\sigma, d\sigma;\omega) \\ \qquad = 1 + U\, D^R(d\sigma;\omega) \\ (\omega + i\eta - \varepsilon_k + \mu)\, G^R(k\sigma, d\sigma;\omega) - t_k^*\, G^R(d\sigma;\omega) = 0 \end{array}\right.$$

$$D^R(d\sigma;\omega) = -i\,\theta(t-t')\left\langle\left\{-[n_{d\uparrow} n_{d\downarrow}, C_{d\sigma}](t),\ C^\dagger_{d\sigma}(t')\right\}\right\rangle$$

ETC: $[n_{d\uparrow} n_{d\downarrow}, C_{d\uparrow}] = -n_{d\downarrow} C_{d\uparrow}$ $\quad\boxed{4.69}$

$$D^R(d\uparrow;\omega) = -i\,\theta(t-t')\left\langle\left\{n_{d\downarrow}^{(t)} C_{d\downarrow}(t),\ C^\dagger_{d\uparrow}(t')\right\}\right\rangle$$

Mean field approximation for Anderson model

$$U n_{d\uparrow} n_{d\downarrow} \xrightarrow{MFA} U\langle n_{d\uparrow}\rangle \underline{n_{d\downarrow}} + U \underline{n_{d\uparrow}} \langle n_{d\downarrow}\rangle - U\langle n_{d\uparrow}\rangle\langle n_{d\downarrow}\rangle$$

* $\boxed{\text{Stoner criterion}}$ * $\Rightarrow \boxed{\rho(\varepsilon_F)\cdot U > 1}$

$$\Leftarrow\ \underline{D^R(d\uparrow;t-t')} = -i\,\theta(t-t')\langle n_{d\downarrow}\rangle \underline{\left\langle\left\{C_{d\uparrow}(t),\ C^\dagger_{d\uparrow}(t')\right\}\right\rangle}$$

energy shift $|\uparrow\rangle$ $\varepsilon_{d\uparrow} \longrightarrow \varepsilon_{d\uparrow} + \underline{\underline{U}}\,\underline{\underline{\langle n_{d\downarrow}\rangle}}$

$$\left[\omega + i\eta - \underline{\varepsilon_d} + \mu - \underline{\underline{U\langle n_{d\downarrow}\rangle}}\right] G^R(d\uparrow,\omega) - \sum_k \frac{|t_k|^2}{\omega - \varepsilon_k + \mu - i\eta}\, G^R(d\uparrow,\omega) = 1 \qquad \boxed{4.70}$$

$$G^R(d\uparrow;\omega) = \frac{1}{\omega - \varepsilon_d + \mu - U\langle n_{d\downarrow}\rangle - \Sigma^R(\omega)}$$

$$\Sigma^R(\omega) = \sum_k \frac{|t_k|^2}{\omega - \varepsilon_k + \mu + i\eta},$$

$$\Sigma^R(\omega) = \int d\varepsilon\, \rho(\varepsilon) \frac{|t(\varepsilon)|^2}{\omega-\varepsilon+\mu+i\eta} \qquad d\vec{k} \to d\varepsilon \cdot \rho(\varepsilon)$$

$$= \mathcal{P}\int d\varepsilon\, \rho(\varepsilon) \frac{|t(\varepsilon)|^2}{\omega-\varepsilon+\mu} - i\pi\, \underline{\rho(\omega+\mu)}\, \underline{|t(\varepsilon+\mu)|^2}$$

assume band-width approximation

$$2\pi\rho(\varepsilon)|t(\varepsilon)|^2 = \begin{cases}\Gamma, & \varepsilon\in[-D,D]\\ 0, & |\varepsilon|>0\end{cases}$$

$$\Sigma^R(\omega) = \frac{\Gamma}{2\pi}\int_{-D}^{D}\frac{d\varepsilon}{\omega-\varepsilon+\mu} - i\Gamma/2$$

↑ shift of energy.

$$A(d\uparrow,\omega) = -2\,\mathrm{Im}\,G^R(d\uparrow,\omega) \qquad \text{Lorentz shape.}$$

$$= \frac{\Gamma}{[\omega-\underset{\uparrow}{\hat{\varepsilon}}+\mu-\underline{U\langle n_{d\downarrow}\rangle}]^2+\Gamma^2/4}$$

$$\langle n_{d\uparrow}\rangle = \int\frac{d\omega}{2\pi}\frac{n_F(\omega)\cdot\Gamma}{[\omega-\hat{\varepsilon}+\mu-U\langle n_{d\downarrow}\rangle]^2+\Gamma^2/4}$$

$T=0$, $\Gamma \ll D$, we have

$$\langle \underline{n_{d\uparrow}}\rangle \simeq \frac{1}{2} - \frac{1}{\pi}\tan^{-1}\left(\frac{\hat{\varepsilon}-\mu+U\langle \underline{n_{d\downarrow}}\rangle}{\Gamma/2}\right)$$

$$\Downarrow$$

$$\begin{cases}\cot(\pi n_\uparrow) = [U n_\downarrow+(\hat{\varepsilon}-\mu)]\big/\Gamma/2\\ \cot(\pi n_\downarrow) = [U n_\uparrow+(\hat{\varepsilon}-\mu)]/\Gamma/2\end{cases} \qquad (4.72)$$

$$\langle n_\uparrow\rangle \neq \langle n_\downarrow\rangle$$

$$\begin{cases} \Delta = \Gamma/2 \\ x = \dfrac{\tilde{\Sigma} - \mu}{U} \\ y = U/\Delta = \dfrac{2U}{\Gamma} \end{cases}$$

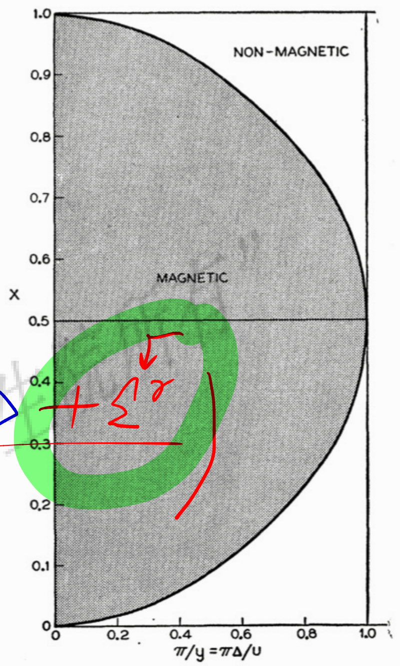


FIG. 4. Regions of magnetic and nonmagnetic behavior. Curve gives x_c vs $\pi/y_c = \pi\Delta/U$.

$$\langle n_{d,\uparrow}\rangle = \frac{1}{2} - \frac{1}{\pi}\tan^{-1}\left(\frac{\varepsilon_d - \mu_{d\uparrow} + \overset{\downarrow\Gamma}{U}\langle n_{d\downarrow}\rangle + \overset{\downarrow}{\Sigma}^r}{\Gamma/2}\right)$$

The two-particle correlation function

retarded charge-charge correlation function

$$\chi^R(rt, r't') = -i\theta(t-t')\langle[\rho(rt), \rho(r't')]\rangle \quad 4.73$$

the random phase approximation: $\underline{\text{RPA}}$

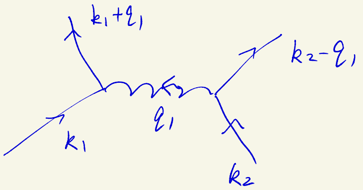
$\underline{1953}$

Bohm, Pines

* derived using Feynman diagrams

* derived using the equation of motion.

$$H = \sum_k \xi_k c_k^\dagger c_k + \frac{1}{2V}\sum_{k_1,k_2,q_1\neq 0} V(q_1)\, c^\dagger_{k_1+q_1} c^\dagger_{k_2-q_1} c_{k_2} c_{k_1}$$



$$\chi^R(\vec{q}, t-t') = -i\theta(t-t')\frac{1}{V}\langle[\rho(\vec{q},t), \rho(-\vec{q},t')]\rangle$$

$$\rho(\vec{q}) = \sum_k c_k^\dagger c_{k+q} = \sum_k c^\dagger_{k-q} c_k \qquad \underline{\chi^R(\vec{q}) = \frac{1}{V}\sum_k \chi^R(k\vec{q})}$$

$$\underline{\chi^R(kq, t-t')} = -i\theta(t-t')\langle[c_k^\dagger c_{k+q}(t), \rho(-\vec{q},t')]\rangle$$

$$i\partial_t \chi^R(kq, t-t') = \delta(t-t') \langle [C_k^+ C_{k+q}(t), \rho(-\vec{q}, t')] \rangle$$

$$-i\theta(t-t')\langle[\,-[H, C_k^+ C_{k+q}](t), \rho(-\vec{q}, t')]\rangle \qquad \boxed{4.74}$$

ETC: $[C_k^+ C_{k+q}, \sum_{k'} C_{k'}^+ C_{k'-q}] = C_k^+ C_k - C_{k+q}^+ C_{k+q}$ ⓑ

$$\begin{cases} [AB, C] = A[B, C] + [A, C]B \\ [A, BC] = [A, B]C + B[A, C] \end{cases}$$

ⓐ

ETC:

$$[H_0, C_k^+ C_{k+q}] = [\sum_k \xi_k C_k^+ C_k, C_k^+ C_{k+q}] = \underline{\underline{[\xi_k - \xi_{k+q}] C_k^+ C_{k+q}}}$$

$$[V_{int}, C_k^+ C_{k+q}] = \frac{1}{2V}\sum_{k_1, k_2, q_1 \neq 0} V(q_1)\left[\overset{④}{C^+_{k_1+q_1}}\underset{③}{C^+_{k_2-q_1}}\overset{②}{C_{k_2}}\underset{①}{C_{k_1}}, \underset{①}{\overset{②}{C_k^+}}\underset{③}{\overset{④}{C_{k+q}}}\right]$$

$$= \frac{1}{2V}\sum_{k', q' \neq 0} V(q')\Big\{\underbrace{C^+_{k+q'} C^+_{k'-q'} C_{k'} C_{k+q}}_{①} + \underbrace{C^+_{k'+q'} C^+_{k-q'} C_{k+q} C_{k'}}_{②}$$

$$- \underbrace{C_k^+ C^+_{k'-q'} C_{k'} C_{k+q-q'}}_{④} - \underbrace{C^+_{k'+q'} C_k^+ C_{k+q+q'} C_{k'}}_{③}\Big\}.$$

Hartree type mean field approximation.

$$C^+_{k+q'} C^+_{k'-q'} C_{k'} C_{k+q} = C^+_{k+q'} C_{k+q} \langle C^+_{k'-q'} C_{k'} \rangle$$

$$+ \langle C^+_{k+q'} C_{k+q} \rangle C^+_{k'-q'} C_{k'} \underset{\star}{-} \langle C^+_{k+q'} C_{k+q} \rangle \langle C^+_{k'-q'} C_{k'} \rangle$$

$$[V_{int}, C_k^+ C_{k+q}] = \frac{1}{2V}\sum_{k'q' \neq 0} V(q')\Big\{C^+_{k+q'} C_{k+q} \overset{\swarrow 0}{\underline{\langle C^+_{k'-q'} C_{k'} \rangle}}.$$

$$n_{k+q}\delta_{qq'} \leftarrow + \langle C^+_{k+\underline{\underline{q'}}} C_{k+\underline{\underline{q}}} \rangle C^+_{k'-q'} C_{k'}$$

$$+\langle c^\dagger_{k-q'}\, c_{k+q}\rangle\, c^\dagger_{k'+q'}\, c_{k'} + c^\dagger_{k-q'}\, c_{k+q}\, \underset{\uparrow\, 0}{\langle c^\dagger_{k'+q'}\, c_{k'}\rangle}$$

$$- c^\dagger_{k'+q'}\, c_{k'}\, \langle c^\dagger_{k}\, c_{k+q+q'}\rangle - \underset{\uparrow\, 0}{\langle c^\dagger_{k'+q'}\cdot c_{k'}\rangle}\, c^\dagger_{k}\, c_{k+q+q'}$$

$$- c^\dagger_{k}\, c_{k+q-q'}\, \underset{\uparrow\, 0}{\langle c^\dagger_{k'-q'}\, c_{k'}\rangle} - \langle c^\dagger_{k}\, c_{k+q-q'}\rangle\, c^\dagger_{k'-q}\, c_{k'} \Big\}$$

$$\underline{\langle c^\dagger_{k}\, c_{k'}\rangle = \langle n_k\rangle\, \delta_{kk'}} \quad (4.76) \qquad = \frac{V(q)}{V}\left[\langle n_{k+q}\rangle - \langle n_k\rangle\right] \underline{\underline{\sum_{k'} c^\dagger_{k'}\, c_{k'+q}}}_{\hat{\rho}(q)}$$

ⓒ (4.75)

$$\Big(\omega + i\eta + \underbrace{\xi_k - \xi_{k+q}}_{ⓐ}\Big)\chi^R(kq,\omega) \quad ⓒ \qquad \boxed{4.76}$$

$$= \underset{\circledast}{-}\left[\langle n_{k+q}\rangle - \langle n_k\rangle\right]\left(1 + \frac{V(q)}{V}\underbrace{\sum_{k'}\chi^R(k'q;\omega)}_{\pi}\right) \quad ⓑ$$

$$\chi^R(q,\omega) = \frac{1}{V}\sum_k \chi^R(kq,\omega) \qquad \boxed{4.77}$$

$$= \frac{1}{V}\sum_k \frac{\langle n_k\rangle - \langle n_{k+q}\rangle}{\omega + \xi_k - \xi_{k+q} + i\eta}\left[1 + \underline{V(q)\chi^R(q,\omega)}\right]$$

free electron gas:

$$\chi_0^R(q,\omega) = \frac{1}{V}\sum_k \frac{\langle n_k\rangle - \langle n_{k+q}\rangle}{\omega + \xi_k - \xi_{k+q} + i\eta}$$

$$\chi^R(q,\omega) = \chi_0^R(q,\omega)\left[1 + V(q)\chi^R(q,\omega)\right]$$

$$\chi^R(q,\omega) = \frac{\chi_0^R(q,\omega)}{1 - V(q)\,\chi_0^R(q,\omega)} \equiv \chi^{R,RPA}(q,\omega)$$

RPA result of polarizability function $\boxed{4.78}$

$$\varepsilon^{RPA}(q,\omega) = \frac{1}{1+V(q)\,\chi^R(q,\omega)} = 1 - V(q)\,\chi_0^R(q,\omega)$$

free-electron gas : $-\mathrm{Im}\,\chi_0^R(q,\omega)$; e-h excitation ;

RPA polarizability function : $\mathrm{Im}\,\chi_0^R(q,\omega) = \delta$

$$-\mathrm{Im}\,\chi^R(q,\omega) = \frac{\delta}{[\underline{1 - V(q)\,\mathrm{Re}\,\chi_0^R(q,\omega)}]^2 + \delta^2} \quad \boxed{\text{pole}}$$

* $1 - V(q)\cdot\mathrm{Re}\,\chi_0^R(q,\omega) = 0$ gives the plasma oscillation mode * $\boxed{4.79}$

<u>plasmon</u>

Imaginary-time Green's function

$$C_{AB}^R(t,t') = -i\theta(t-t')\,\langle [A(t), B(t')]_{B,F} \rangle$$

$$C_{AB}(t,t') = -\langle A(t)\,B(t') \rangle$$

$$C^R = i\theta(t-t')\,[C_{AB} \mp C_{BA}]$$

$$C_{AB}(t,t') = -\frac{1}{Z}\,\mathrm{Tr}\left[e^{-\beta H} A(t)\, B(t')\right]$$

↑ Heisenberg picture.

$$\updownarrow$$

ETC: $$\boxed{|\hat{\psi}(t)\rangle = \hat{U}(t,t_0)|\hat{\psi}(t_0)\rangle}$$

$$C_{AB}(t,t') = -\frac{1}{Z}\,\mathrm{Tr}\left[e^{-\beta H}\,\hat{U}(0,t)\,\hat{A}(t)\,\hat{U}(t,t')\,\hat{B}(t')\,\hat{U}(t',0)\right]$$

↑ Interaction picture

$$\boxed{\hat{U}(t,t_0) = e^{iH_0 t}\, e^{-iH(t-t_0)}\, e^{-iH_0 t_0}} \quad \circledast$$

$$i\partial_t|\hat{\psi}(t)\rangle = \hat{V}(t)\,|\hat{\psi}(t)\rangle$$

$$|\hat{\psi}(t)\rangle = \hat{U}(t,t_0)\,|\hat{\psi}(t_0)\rangle$$

$$i\partial_t\,\hat{U}(t,t_0) = \hat{V}(t)\,\hat{U}(t,t_0) \qquad \hat{U}(t_0,t_0) = 1$$

$$\hat{U}(t,t_0) = T_t\left[e^{-i\int_{t_0}^{t} dt'\,\hat{V}(t')}\right]$$

Imaginary time H-picture.

$$A(\tau) = e^{\tau H}\, A\, e^{-\tau H} \qquad \tau \quad \text{Imaginary time}$$

$$\tau \to it$$

$$A(t) = e^{itH} A e^{-itH}$$

Imaginary time I-picture

$$\hat{A}(\tau) = e^{\tau H_0}\, A\, e^{-\tau H_0}$$

$$A(\tau)\,B(\tau') = \hat{U}(0,\tau)\,\underline{\hat{A}(\tau)\,\hat{U}(\tau,\tau')\,\hat{B}(\tau')\,\hat{U}(\tau',0)}$$

(4.80)

$$\hat{U}(\tau,\tau'')\,\hat{U}(\tau'',\tau') = \hat{U}(\tau,\tau')$$

$$\partial_\tau \hat{U}(\tau,\tau') = -\hat{V}(\tau)\,\hat{U}(\tau,\tau') \Leftarrow$$

$$\hat{U}(\tau,\tau') = T_\tau\left\{\exp\left[-\int_{\tau'}^{\tau} d\tau_1\, \hat{V}(\tau_1)\right]\right\}$$

$$\hat{U}(\tau,\tau') = e^{\tau H_0}\, e^{-(\tau-\tau')H}\, e^{-\tau' H_0} \qquad 4.81$$

$$e^{-\beta H} = \underline{\underline{e^{-\beta H_0}}}\ \underline{\underline{\hat{U}(\beta,0)}} \qquad \begin{cases}\tau=\beta\\ \tau'=0\end{cases}$$

Definition of Matsubara Green's function.

$$C_{AB}(\tau,\tau') \equiv -\langle \underset{\uparrow}{T_\tau}[A(\tau)B(\tau')]\underset{\uparrow}{\rangle}$$

$$T_\tau[A(\tau)B(\tau')] = \theta(\tau-\tau')A(\tau)B(\tau') \pm \theta(\tau'-\tau)B(\tau')A(\tau)$$

(4.82)

+ Bosons

− Fermions

① $\boxed{\tau>\tau'}$

$$C_{AB}(\tau,\tau') = -\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}e^{\tau H}\underset{\uparrow}{A}e^{-\tau H}e^{\tau' H}Be^{-\tau' H}\right]$$

S-picture

$$= -\frac{1}{Z}\mathrm{Tr}\left[\underline{\underline{e^{-\beta H}\,e^{(\tau-\tau')H}}}\,A\,e^{-(\tau-\tau')H}\underset{\uparrow}{B}\right]$$

$$= C_{AB}(\underline{\underline{\tau-\tau'}})$$

$\boxed{(\tau-\tau') > \beta}$

② $-\beta < \underline{\underline{\tau-\tau'}} < \beta$

(4.83)

$C_{AB}(\tau)$; $\underline{\tau\in(-\beta,\beta)}$

③

$$\tau<0\;;\qquad C_{AB}(\tau+\beta) = \underset{\text{fermion}}{\overset{+\ \text{Boson}}{\pm}} C_{AB}(\tau)$$

$$C_{AB}(\tau+\beta) = \frac{-1}{Z}\mathrm{Tr}\left[e^{-\beta H} e^{(\tau+\beta)H} \underset{\uparrow\ \text{S-picture}}{A} e^{-(\tau+\beta)H} B\right]$$

$$= \frac{-1}{Z}\mathrm{Tr}\left[B\cdot e^{\tau H} A e^{-\tau H} e^{-\beta H}\right]$$

$$= \frac{-1}{Z}\mathrm{Tr}\left[e^{-\beta H} B\, \underbrace{e^{\tau H} A e^{-\tau H}}\right]$$

$$\boxed{\tau<0} \swarrow \qquad = \pm\left\langle T_\tau\left(A(\tau)\, B\right)\right\rangle$$

$$= \pm C_{AB}(\tau) \qquad (484)$$

Fourier transformation of Matasubara Green's function.

$$C_{AB}(n) = \frac{1}{2}\int_{-\beta}^{\beta} d\tau\, e^{i\pi n\tau/\beta} C_{AB}(\tau)$$

$$C_{AB}(\tau) = \frac{1}{\beta}\sum_{n=-\infty}^{\infty} e^{-i\pi n\tau/\beta} C_{AB}(n)$$

$$C_{AB}(n) = \frac{1}{2}\int_0^{\beta} d\tau\, e^{i\pi n\tau/\beta} C_{AB}(\tau) + \underline{\frac{1}{2}\int_{-\beta}^{0} d\tau\, e^{i\pi n\overset{\tau-\beta}{\tau}/\beta} C_{AB}(\tau)}$$

$$\frac{1}{2}\int_0^{\beta} d\tau\, e^{i\pi n(\tau-\beta)/\beta}\, \underline{C_{AB}(\tau-\beta)}$$

$$\Downarrow$$

$$C_{AB}(\tau-\beta) = \pm C_{AB}(\tau)$$

$$= \frac{1}{2}\left(1 \pm e^{-i\pi n}\right)\int_0^{\infty} d\tau\, e^{i\pi n\tau/\beta} C_{AB}(\tau)$$

$$e^{-i\pi n} \Rightarrow \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$$