Lesson 5

$$\dot{\hat{S}}_i^+ \simeq -i\,|J|\,\hbar S \sum_\delta \left[\hat{S}_i^+ - \hat{S}_{i+\delta}^+ \right]$$

Fourier transformation

$$-i\omega_q = -i\,|J|\,\hbar S \cdot Z\,(1-\gamma_q)$$

$$\gamma_q = \frac{1}{Z}\sum_\delta e^{i\vec{k}\cdot\vec{\delta}}$$

$$\omega_q = |J|\cdot\hbar S\cdot Z\,(1-\gamma_q)$$

$$\boxed{\text{AFM spin wave.}}$$

$$\mathcal{H} = -J\sum_{\langle i,j\rangle} \hat{S}_i\cdot\hat{S}_j$$

$J<0$

$$= -J\sum_{\langle i,j\rangle}\left[\hat{S}_i^z\hat{S}_j^z + \tfrac{1}{2}\left(\hat{S}_i^+\hat{S}_j^- + \hat{S}_i^-\hat{S}_j^+\right)\right]$$

Néel state a b a b

$|\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\cdots\rangle$

$i\ j$ ... $s\ \ -s$ ... $\uparrow\downarrow$ ($i\ j$) ... $S-1\ \ -S+1$ $\downarrow\uparrow$ ($i\ j$) ...

$\hat{S}_i^z\hat{S}_j^z$ $\qquad$ $\hat{S}_i^+\hat{S}_j^-$ $\to 0$ $\qquad$ $S_i^- S_j^+$ ☆

Née state : low energy state ; but Néel is not the eigenstate.

Anderson 1952.

* Q1 : Ground state
* Q2 : excitation state

} 2nd quantization

* H-P transformation
* Bogoliubov 变换

* 张礼 7.1 ; 10.1 ~ 10.4 ; ← * BEC

* 李正中 3.5

* Khomskii chapter 5

we consider the Néel state as a reference.

$|0, 0, \cdots, 0\rangle$ Néel $\begin{array}{cccc} A & B & A & B \\ |S & \underline{-S} & S & -S \cdots\rangle \end{array}$

any state $|n_1, n_2 \cdots, n_N\rangle$

sublattice A $\underline{\underline{n_i}} = S - S_i^z$

Sublattice B : $\underline{\underline{n_j}} = S_j - (-S) = S_j^z + S$

particle number is not conserved.

H-P transformation $S_i^+ S_j^- \Rightarrow \underline{\underline{a_i b_j}}$ $S_i^- S_j^+ \Rightarrow \underline{\underline{a_i^+ b_j^+}}$

for A : $\begin{cases} S_i^+ = \sqrt{2S}\, a_i \\ S_i^- = \sqrt{2S}\, a_i^+ \\ S_i^z = S - a_i^+ a_i \Rightarrow a_i^+ a_i = S - S_i^z \end{cases}$

$\underline{\sqrt{2S - a_i^+ a_i}} \simeq \sqrt{2S}$

$\Rightarrow$ for B: $\begin{cases} S_j^+ = \sqrt{2S}\, b_j^+ \\ S_j^- = \sqrt{2S}\, b_j \\ S_i^z = b_j^+ b_j - S \Rightarrow b_j^+ b_j = S_i^z + S \end{cases}$

ETC: $\begin{cases} [a_i, a_{i'}^+] = \delta_{ii'} \\ [b_j, b_{j'}^+] = \delta_{jj'} \end{cases}$

Fourier transformation:

$$c_k = \frac{1}{\sqrt{N}} \sum_i e^{i\vec{k}\cdot\vec{R}_i} a_i$$

$$d_k = \frac{1}{\sqrt{N}} \sum_j e^{i\vec{k}\cdot\vec{R}_j} b_j$$

ETC: $\quad z$: number of nearest neighbour

$$\mathcal{H} = -2NJzS + 2JzS\sum_k \Big[\underbrace{\gamma_k\,(c_k^+ d_k^+ + c_k d_k)}_{\text{hopping. N.N.}} + \underbrace{(c_k^+ c_k + d_k^+ d_k)}_{\text{on-site}} \Big]$$

$$\gamma_k = \frac{1}{z}\sum_\delta e^{i\vec{k}\cdot\vec{R}_\delta}$$

Bogoliubov transform:

$$\star\ \begin{cases} \alpha_k = u_k c_k - v_k d_k^+ \\ \alpha_k^+ = u_k^* c_k^+ - v_k^* d_k \end{cases} \qquad \begin{cases} \beta_k^+ = u_k^* d_k - v_k^* c_k^+ \\ \beta_k = u_k d_k^+ - v_k c_k \end{cases}$$

$$\begin{cases} [\alpha_k, \alpha_{k'}^+] = [\beta_k, \beta_{k'}^+] = \delta_{kk'} \\ [\alpha_k, \alpha_{k'}] = [\beta_k, \beta_{k'}] = [\alpha_k, \beta_{k'}^+] = [\alpha_k^+, \beta_{k'}] = 0 \end{cases}$$

ETC: $\star\ |u_k|^2 - |v_k|^2 = 1 \Leftarrow$ Bose B-T

$$\boxed{|u_k|^2 + |v_k|^2 = 1} \Leftarrow \text{Fermi B-T}$$

guess: $\mathcal{H} = \text{const} + \sum_k \lambda_k(\alpha_k^+\alpha_k + \beta_k^+\beta_k) \Rightarrow$

original: $\mathcal{H} = -2NJzS + 2JzS\sum_k [\gamma_k(c^+d^+ + cd) + c^+c + d^+d]$

$$\omega_0 = 2JzS\ ; \quad \omega_1 = 2JzS\gamma_k$$

$$\begin{cases} [\alpha_k^+, \mathcal{H}] = -\lambda_k \alpha_k^+ \\ [\alpha_k, \mathcal{H}] = \lambda_k \alpha_k \end{cases} \Rightarrow$$

$$[\alpha_k^+, H] = -u_k(\omega_0 c_k^+ + \omega_1 d_k) - v_k(\omega_0 d_k + \omega_1 c_k^+)$$

$$\Updownarrow$$

$$[\alpha_k^+, H] = -\lambda_k \alpha_k^+ = -\lambda_k[u_k c_k^+ - v_k d_k]$$

$$\begin{cases} \omega_0 u_k + \omega_1 v_k = \lambda_k u_k \\ \omega_1 u_k + \omega_0 v_k = -\lambda_k u_k \end{cases} \Rightarrow \begin{bmatrix} \omega_0 - \lambda_k & \omega_1 \\ \omega_1 & \omega_0 + \lambda_k \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix} = 0 \quad (A)$$

$$\mathrm{Det}\begin{vmatrix} \omega_0-\lambda_k & \omega_1 \\ \omega_1 & \omega_0+\lambda_k \end{vmatrix} = 0 \qquad \lambda_k^2 = \omega_0^2 - \omega_1^2$$

$$\lambda_k \geq 0$$

$$\lambda_k = \sqrt{\omega_0^2-\omega_1^2} = 2JZS\sqrt{1-\gamma_k^2} \qquad (C)$$

$$U_k^2 - V_k^2 = 1 \quad (B)$$

From (A) and (B).

$$\begin{cases} U_k^2 = \frac{1}{2}\left[\frac{1}{\sqrt{1-\gamma_k^2}} + 1\right] \\ V_k^2 = \frac{1}{2}\left[\frac{1}{\sqrt{1-\gamma_k^2}} - 1\right] \end{cases} \qquad (D)$$

From the Bogoliubov tranform, we have the final result:

$$\mathcal{H} = \underbrace{-2NJZS(S+1)}_{\text{Néel}} + \sum_k 2JZS\sqrt{1-\gamma_k^2}\left(\alpha_k^+\alpha_k + \beta_k^+\beta_k + 1\right)$$

ground state energy: $E_g = -2NJZS(S+1) + \sum_k 2JZS\sqrt{1-\gamma_k^2}$

two branches of degenerate energy spectrum for AFM spin wave.

AFM spin wave: $\hbar\omega_k = 2JZS\sqrt{1-\gamma_k^2}$

$k \to 0$; $\hbar\omega_k \propto \sqrt{(1-\gamma_k)(1+\gamma_k)} \propto k$

FM spin wave:

$$\hbar\omega_k = 2JZS(1-\gamma_k)$$

$k\to 0$; $1-\gamma_k \propto k^2$

$k\to 0$; $1+\gamma_k \simeq 1$

$\hbar\omega_k \propto k^2$

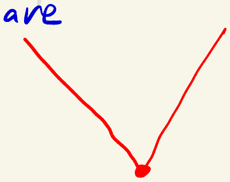
Goldstone Mode: Gapless.

Massless

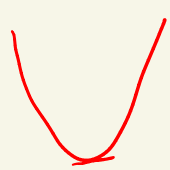
AFM- spin wave

gapless

massless



break rotation symmetry.



FM spin wave.

gapless

$\Rightarrow$ gapless excitation modes.

$\forall k$; $\alpha_k|\rangle = 0 \qquad |\rangle$ ground state.

$\beta_k|\rangle = 0$

Q: change from Néel　　　　　　　　　　　　Bose expectation value

$$\Delta M_s = \sum_{i,j} (a_i^\dagger a_i + b_j^\dagger b_j) = \sum_k \left[u_k^2 \langle \alpha_k^\dagger \alpha_k \rangle + v_k^2 \langle \alpha_k \alpha_k^\dagger \rangle \right]$$

2分 下周三前发给我。

① $T = 0$; $d \le 1$, ΔM_s diverges.　　T=0; 1D no long-range AFM

② $T > 0$; $d \le 2$; ΔM_s diverges.　　T>0; 2D; no long-range AFM

$\boxed{\text{Hohenberg - Mermin - Wagner theorem.}}$

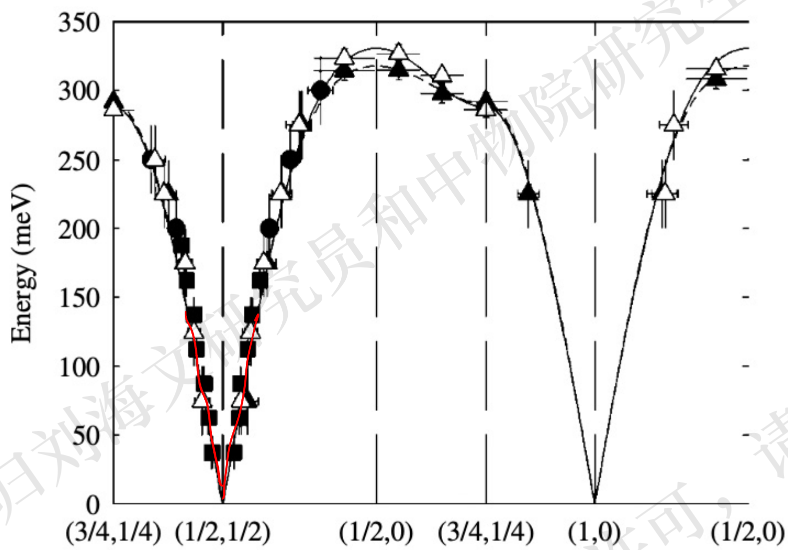


Figure 2.14 Experimentally obtained spin-wave dispersion of the high-T_c parent compound $LaCuO_4$ – a prominent spin 1/2 antiferromagnet. Reprinted with permission from R. Coldea S. M. Hayder, G. Aeppli, *et al.*, Spin waves and electronic excitations in La_2CuO_4, *Phys. Rev. Lett.* **86** (2001), 5377–80. Copyright (2001) by the American Physical Society.

* 2D magnetic system ; 2D XY model : BKT transition.
* 1D magnetic chain ; 张礼书 chapter 7.
* Ising model : 高等统计物理.
* disordered magnetic system. *

Jordan-Wigner transformation.

Quantum spin ½ mapping to fermion operator.

$$|\uparrow\rangle \equiv |1\rangle = f^+|0\rangle \qquad \begin{cases} \hat{S}^+ = f^+ \\ \hat{S}^- = f^- \\ \hat{S}^z = f^+ f - 1/2 \end{cases}$$

$$|\downarrow\rangle \equiv |0\rangle = f|1\rangle$$

ETC : $[\hat{S}^+, \hat{S}^-] = 2\hat{S}^z$

$i \neq j$. different site : $S_i^+ S_j^+ = S_j^+ S_i^+$

fermion operator : $f_i^+ f_j^+ = - f_j^+ f_i^+$

Jordan – Wigner transformation $\Longrightarrow$ change the topological properties

1D $\boxed{\text{non-local nonlinear}}$ transformation

1D Heisenberg chain:

$$\begin{cases} \hat{S}_l^+ = f_l^+ e^{i\pi \sum_{j<l} \hat{n}_j} \\ \hat{S}_l^- = e^{-i\pi \sum_{j<l} \hat{n}_j} f_l \\ \hat{S}_l^z = f_l^+ f_l - 1/2 \end{cases}$$

$$\hat{H} = -\sum_n \left[J_z \hat{S}_n^z \hat{S}_{n+1}^z + \frac{J_{xy}}{2} \left(\hat{S}_n^+ \hat{S}_{n+1}^- + \hat{S}_n^- \hat{S}_{n+1}^+ \right) \right]$$

$$\Downarrow$$

$$\hat{H} = \sum_n \left[\frac{J_{xy}}{2} \left(f_n^+ f_{n+1} + h.c. \right) + J_z \left(\frac{1}{4} - f_n^+ f_n + f_n^+ f_n f_{n+1}^+ f_{n+1} \right) \right]$$

verify : $\hat{S}_m^+ S_{m+1}^- = f_m^+ f_{m+1}$;

Thus for 1D XY model :

$$J_z = 0 ; \qquad \hat{H} = -\frac{J_{xy}}{2} \sum_n \left(f_n^+ f_{n+1} + h.c. \right)$$

non-interacting spinless fermion

Weakly Interacting Bose system.

① non-interacting Bosons ; Bose-Einstein Condensation.

$$\mathcal{H}_B = \sum_{\vec{k}} \left(\underline{\frac{\hbar^2 k^2}{2m}} - \mu \right) \hat{n}_{\vec{k}}$$

μ ← chemical potential

$\hat{n}_k = b_k^+ b_k$

$[b_k, b_{k'}^+] = \delta_{kk'}$

$[b_k, b_{k'}] = [b_k^+, b_{k'}^+] = 0$

$$|N\rangle = \frac{1}{\sqrt{N!}} (b_0^+)^N |0\rangle \Leftarrow \text{ground state}$$

b_0: $\boxed{k=0}$

coherent state. ⇐ ground state.

$$\Rightarrow b_0 |\alpha\rangle\ \alpha|\alpha\rangle$$

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha b_0^+} |0\rangle = e^{-|\alpha|^2/2} \sum_{N=0}^{\infty} \frac{\alpha^N}{\sqrt{N!}} |N\rangle$$

$\langle\alpha|\hat{n}|\alpha\rangle$ not a fixed number.

$$U|\alpha\rangle \Rightarrow |\alpha^*\rangle \qquad b_0|\alpha^*\rangle = \alpha^*|\alpha\rangle$$

$$\alpha^* = e^{i\theta}\alpha$$

$\Big\{$ Coherent state $|\alpha\rangle$ breaks the $U(1)$ symmetry.
$\phantom{\Big\{}$ H_B satisfies $U(1)$ symmetry.

Spontaneous symmetry breaking.

* BEC in non-interacting Bose system.

* excitation : $\varepsilon_{\vec{k}} = \dfrac{\hbar^2 k^2}{2m}$ quadratic dispersion. ✗

✗ Not a superfluid ✗

② off-diagonal long-range order.

definition of BEC momentum $k = 0$

$$n_0 \equiv \frac{1}{V} \langle a_0^+ a_0 \rangle > 0 \Leftarrow$$ Penrose – Onsager 1950

long-range correlation in real space.

$$\hat{\varphi}(\vec{r}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} a_{\vec{k}} \qquad \int_{-\infty}^{\infty} \sin kx \, dx \rightsquigarrow 0$$

$$\langle \hat{\varphi}^+(\vec{r}) \varphi(\vec{r}') \rangle = \frac{1}{V} \langle a_0^+ a_0 \rangle + \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}(\vec{r}-\vec{r}')} \langle a_k^+ a_k \rangle$$

$\longrightarrow 0$

$$\boxed{\langle \hat{\varphi}^+(\vec{r}) \hat{\varphi}(\vec{r}') \rangle \longrightarrow n_0} \ (*)$$ irrelevant to $|r - r'|$

off-diagonal long-Range order C. N. Yang.

non-local

P. W. Anderson volume V

consider a box of N particles in $k=0$ state,

$$a_0 = \frac{1}{\sqrt{V}} \sum_i \varphi_i$$

$$\langle a_0^+ a_0 \rangle = N \Rightarrow \langle a_0^+ \rangle = \langle a_0 \rangle = \sqrt{N}$$

$$N = \langle a_0^+ a_0 \rangle = \frac{1}{V} \sum_i \varphi_i^+ \varphi_i + \frac{1}{V} \sum_{i \neq j} \varphi_i^+ \varphi_j$$

$$= \frac{N}{V} + \frac{1}{V} \sum_{i \neq j} \varphi_i^+ \varphi_j$$

$\varphi_i^+ \varphi_j$ must be long-range correlated.

off-diagonal long-rang order $\longleftarrow$ BEC

diagonal long-range order $\longleftarrow$ lattice.

ODLRO + DLRO $\longleftarrow$ supersolid

Chemical potential $\mu \to 0$; approaching BEC $\quad \beta = \frac{1}{k_B T}$

$$\frac{N}{V} = \rho = \frac{\sum_k \langle n_k \rangle}{V} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1}$$

$\mu = 0$

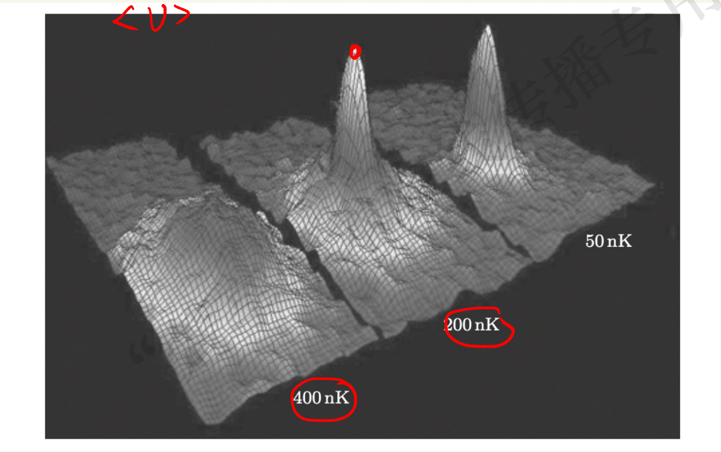
$$\varepsilon_k = \frac{\hbar^2 k^2}{2m} \; ; \quad \text{thermal wave length} \quad \lambda = \sqrt{\frac{\beta \hbar^2}{2m}}$$

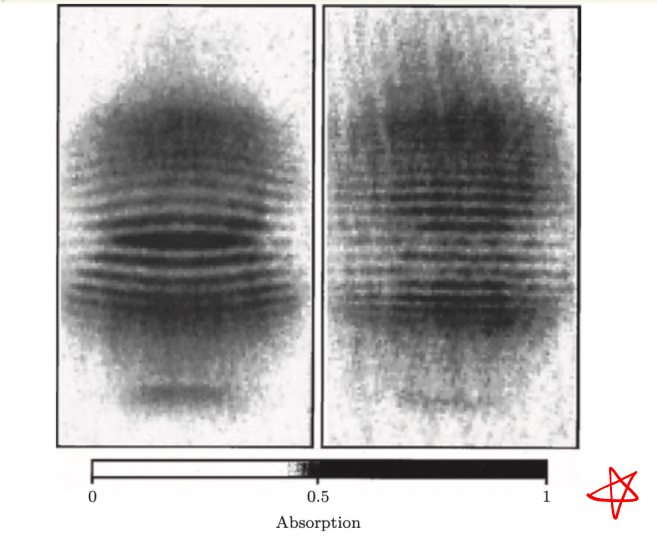
$$\longrightarrow \rho = \frac{\lambda^{-3}}{2\pi^2} \int_0^\infty \frac{x^2\, dx}{e^{x^2} - 1} = \lambda^{-3} \cdot \frac{\zeta(3/2)}{8\pi^{3/2}}$$

given

$$\text{Critical } T_c = \left[\frac{\rho}{\zeta(3/2)}\right]^{3/2} \frac{2\pi \hbar^2}{k_B m}$$

when $T < T_c$; BEC





1 ~ 10 mK solid system

BEC

Interference between two BEC

1995

Weakly Interacting Bosons and Bogoliubov Theory.

Interacting Bosons

$$\mathcal{H} = \hat{T} + \hat{V} = \sum_i \frac{p_i^2}{2m} + \sum_{i<j} V(\vec{r}_i - \vec{r}_j)$$

second quantization

$$\mathcal{H} = \sum_k \varepsilon_k b_k^+ b_k + \frac{1}{2V} \sum_{k_1 k_2 k_3} \underset{\uparrow V_0}{V_{k_1-k_3}} \underline{b_{k_1}^+ b_{k_2}^+ b_{k_3} b_{k_1+k_2-k_3}}$$

$$V_k = \int d^3r\, e^{-i\vec{k}\cdot\vec{r}} V(\vec{r})$$

* Huang kerson ; Yang ; Lee ; psuedo-potential

ultra-local approximation ; $V(\vec{r}) = V_0 \delta(\vec{r})$

$$U_k = V_0$$

approximation :

leading order = $b_0^+ b_0$; high order ; $b_k^+ b_k$

$$\hat{V} \simeq \frac{V_0}{2V}\left[b_0^+ b_0^+ b_0 b_0 + 4 b_0^+ b_0 \sum_{k\neq 0} b_k^+ b_k + \sum_{k\neq 0} b_0^+ b_0^+ b_k b_{-k} + b_0 b_0 b_k^+ b_{-k}^+\right]$$

classical number

$$\langle b_0^+ b_0 \rangle + \sum_{\vec{k}\neq 0} \langle b_k^+ b_k \rangle = N \qquad \langle b_0 \rangle = \langle b_0^+ \rangle = \sqrt{N}$$

approximation $\langle b_0^+ b_0 \rangle \simeq N$

$$b_0 b_0^+ - b_0^+ b_0 = 1$$

$$\hat{V} \simeq \frac{N(N-1)V_0}{2V} + \frac{N\cdot V_0}{V}\sum_{k\neq 0}\left[b_k^+ b_k + \frac{1}{2N}\left(\underline{b_0 b_0}^{\,N} b_k^+ b_{-k}^+ + \underline{b_0^+ b_0^+}^{\,N} b_k b_{-k}\right)\right]$$

$$\boxed{n = \frac{N}{V}}$$

$$H_B = \frac{N(N-1)V_0}{2V} + \sum_{k\neq 0}\left[(\varepsilon_k + nV_0) b_k^+ b_k + \frac{nV_0}{2}(b_k^+ b_{-k}^+ + b_k b_{-k})\right]$$

$$\varepsilon_k = \varepsilon_{-k}$$

$$= \frac{N(N-1)V_0}{2V} + \sum_{k>0} \Big[(\varepsilon_k + nV_0)(b_k^+ b_k + b_{-k}^+ b_{-k})$$

Bogoliubov transformation: $$+ \; nV_0 (b_k^+ b_{-k}^+ + b_k b_{-k}) \Big]$$

We have known for AFM spin wave Bogoliubov transformation

$$\mathcal{H} = \text{const} + \sum_k \lambda_k (\alpha_k^+ \alpha_k + \beta_k^+ \beta_k) \Rightarrow$$

$$\mathcal{H} = -2NJzS + 2JzS \sum_k [\gamma_k (c^+ d^+ + cd) + c^+ c + d^+ d]$$

$$\lambda_k = \sqrt{1 - \gamma_k^2} \cdot 2JzS$$

$$H_B = \frac{N(N-1)V_0}{2V} + \sum_{k>0} \sqrt{(\varepsilon_k + nV_0)^2 - (nV_0)^2} \left(\alpha_k^+ \alpha_k + \alpha_{-k}^+ \alpha_{-k} \right)$$

excitation

$$= \frac{N(N-1)V_0}{2V} + \sum_{k \neq 0} \sqrt{\varepsilon_k (\varepsilon_k + 2nV_0)} \; \alpha_k^+ \alpha_k \quad + \text{const}$$

$\boxed{\text{Bogoliubov transformation}}$ Identical to AFM spin wave.

① excitation spectrum. $k \to 0$; $\varepsilon_k \simeq \sqrt{\varepsilon_k \cdot 2nV_0} = \hbar v_s \cdot k$

linear spectrum; $v_s = \sqrt{nV_0/m}$; sound velocity.

② ground state energy. $\dfrac{N(N-1)V_0}{2V}$

Compressibility. $\kappa = \dfrac{1}{n^2} \left(\dfrac{\partial n}{\partial \mu} \right)$ noninteracting. $\kappa = \infty$

interacting boson: $\kappa = \dfrac{1}{n^2 V_0}$ $\quad v_s = \sqrt{\dfrac{1}{\kappa \rho}} \to 0$

$\rho = n \cdot m$

$$E = \frac{N(N-1)V_0}{V} \qquad \mu = \frac{\Delta E}{\Delta N} = E(N+1) - E(N) = nV_0$$

Sound velocity: $v_s = \sqrt{\dfrac{1}{\kappa \rho}} = \sqrt{nV_0/m}$

Stability of BEC
- → dimensionality. ✓
- → topological excitation ✓
- → superfluid velocity. ✓

① dimensionality:

excitation Number

$$\Delta N = \sum_{k\neq 0} \langle b_k^{+} b_k \rangle \overset{\text{ETC}}{=\!=} \sum_{k\neq 0} \left(U_k^2 \langle \alpha_k^{+}\alpha_k \rangle + V_k^2 \langle \alpha_k \alpha_k^{+} \rangle \right)$$

① $T = 0$; $\Delta N = \sum_{k\neq 0} V_k^2$; $d \leq 1$; ΔN diverge

② $T > 0$; $\Delta N = \sum_{k\neq 0} \left[(U_k^2 + V_k^2) \langle \alpha_k^{+}\alpha_k \rangle \right] + \sum_{k\neq 0} V_k^2$

Bose distribution

$T > 0$; $d \leq 2$; ΔN diverge.

BEC; $T > 0$; for $d \leq 2$ BEC is unstable

$T = 0$; for $d \leq 1$, BEC is unstable

* Hohenberg - Mermin - Wagner theorem. *

② superfluid velocity.

$$\vec{v} = \frac{\hbar \vec{q}}{m}$$

noninteracting: $\frac{\hbar^2 q^2}{2m}$ ← Boson Impurity scatters → $q = 0$

Bogoliubov state:

Impurity scatters. $\frac{\hbar^2 q^2}{2m}$ $q \neq 0$ ① ⟶ $q = 0$ ②

kinetic energy $\quad -\dfrac{\hbar^2 q^2}{2m}$

potential energy: $\quad \boxed{\dfrac{N V_0}{V} = n V_0}$

Stable creteria: $\quad -\dfrac{\hbar^2 q^2}{2m} + n V_0 > 0$

$$\hbar q < \sqrt{2}\sqrt{m n V_0}$$

rough estimation: $\quad v \simeq \dfrac{\hbar q}{m} = \sqrt{2}\sqrt{\dfrac{n V_0}{m}} \simeq \sqrt{2}\, V_s$

* BEC; Local order Parameter [Landau-Ginzberg]

$$\langle \varphi^{+}(r) \rangle = \sqrt{\rho(r)}\, e^{i\phi(r)} \qquad \phi(r) \to \phi$$

$$\vec{j}(\vec{r}) = \frac{\hbar}{2m}\left\{ \psi^{*}(r)(-i\nabla)\psi(\vec{r}) + \psi(\vec{r})\left[i\nabla \psi^{*}(\vec{r})\right]\right\}$$

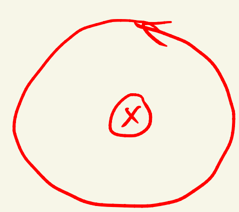
$$= \frac{\hbar}{m}\rho(r)\, \nabla\phi(r)$$

phase gradient is the SF velocity.

③ topological excitation. $\quad \vec{j} = \rho(r)\cdot \vec{V_s}$

$$\begin{cases} \vec{V_s} = \dfrac{\hbar}{m}\nabla \boxed{\phi} \\ \boxed{\nabla \times \vec{V_s} = 0} \end{cases} \quad \to \; ?$$

$\Delta\phi = 2\pi \cdot n$



flux

$\Downarrow$

vortice.

$$\oint \nabla \times \vec{V_s} = \oint \vec{V_s}\cdot d\vec{l} = \frac{\hbar}{m} 2\pi \cdot n$$

$n \in \mathbb{Z}$ $\quad$ $d\vec{l}$ $\to$ change of ϕ

eigenfunction of SF is single-value.

influence stability

n is the flux number $\quad$ * flux - flux interaction.

2D system.