Lesson 4

$$U_{iiii} = U/2 \quad \sum_{i\sigma\sigma'} U_{iiii}\, a^+_{i\sigma} a^+_{i\sigma'} a_{i\sigma'} a_{i\sigma}$$

$$= U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}$$

Hubbard Model

$$\hat{H} = -t \sum_{\langle ij \rangle} a^+_{i\sigma} a_{j\sigma} + U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}$$

bandwidth t Coulomb interaction U

half-filled occupancy is 1

$U/t \ll 1$ $U/t > 1$

weak metal

strong interaction

Mott insulator

① U/t ; ② T/t ③ filling fraction n

strong-coupling limit.

two sites

$U \gg t$

Anderson perturbat.

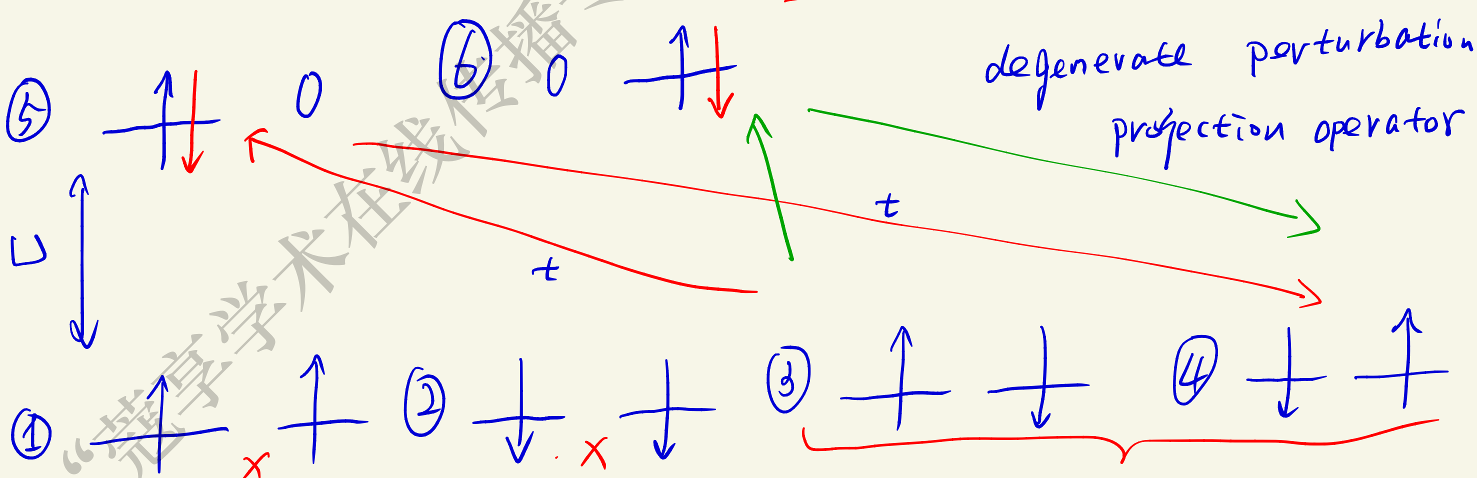
super-exchange

$$H_1 = -t \sum_{\langle ij \rangle} a^+_{i\sigma} a_{j\sigma}$$

$$\boxed{H_0 = U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}}$$

degenerate perturbation

projection operator



$$H_2 = -\frac{4t^2}{U}$$

$$H = J \sum_{\langle ij \rangle} \hat{S}_i \cdot \hat{S}_j \qquad J = \frac{4t^2}{U}$$

Ferromagentic spin wave.

$$= (\hat{S}_i^z \cdot \hat{S}_j^z + \hat{S}_i^x \cdot \hat{S}_j^x + \hat{S}_i^y \cdot \hat{S}_j^y)$$

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \hat{S}_i \cdot \hat{S}_j \quad \text{Spin } S \quad \begin{cases} \hat{S}_i^+ = \hat{S}_i^x + i\hat{S}_i^y \\ \hat{S}_i^- = \hat{S}_i^x - i\hat{S}_i^y \end{cases}$$

$$J > 0 \quad |S,S\rangle = |\uparrow\uparrow \cdots \uparrow\rangle \quad |\downarrow\downarrow \cdots \downarrow\rangle$$

Spin: S $\quad 2S+1$ degeneracy.

spin 1/2

$$\hat{S}_i^+ = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \qquad \hat{S}_i^- = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

$$\Phi_m = S_j^- |S,S\rangle = |\uparrow\uparrow \cdots \downarrow \uparrow\uparrow\uparrow\rangle$$

$$[H, S^-] = 0 \quad * \text{ state}$$

$$S^- \equiv \sum_{j=1}^{N} S_j^- \quad \circledast\, |\alpha\rangle = \exp[\,i\vec{\alpha}\cdot\hat{S}^-]\,|S,S\rangle \text{ ground } \circledast$$

number of N.N.

ground state $\quad E_0 = -J \cdot S^2 \cdot Z \cdot N$

ETC.

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} [S_i^z S_j^z + \tfrac{1}{2}(S_i^+ S_j^- + S_i^- S_j^+)]$$

$$[S_m^+, S_n^-] = 2 S_m^z \delta_{mn}$$

$$[S_m^-, S_n^z] = S_m^- \delta_{mn}$$

$$[S_m^+, S_n^z] = - S_m^+ \delta_{mn}$$

← Sakurai Modern QM. Angular Momentum.

equation of motion.

ETC

$$\boxed{\mathcal{H}\,\Phi_m = E_0 \Phi_m + 2JS \sum_{\delta} (\Phi_m - \Phi_{m+\delta})}$$

$$\Phi_k = \sum_m e^{ik\cdot R_m} \Phi_m$$

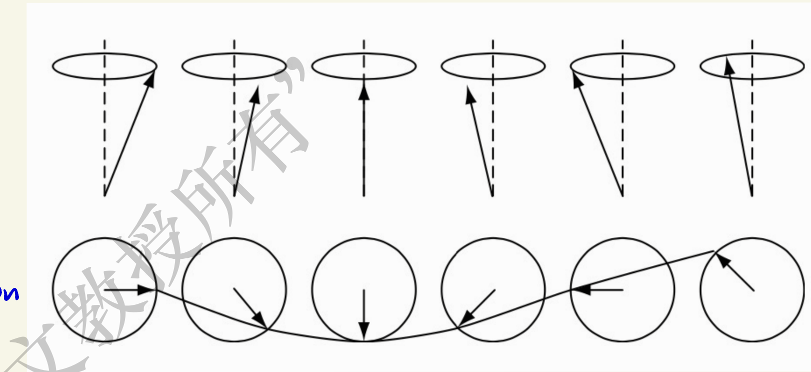
$$\mathcal{H}\,\Phi_k = E_0 + 2JSZ\,[1 - \underbrace{\frac{1}{Z}\sum_{\delta} e^{ik\cdot R_\delta}}_{\gamma_k}]$$

$$E_k = E_0 + \underline{2JSZ\,[1-\gamma_k]}$$

$$\gamma_k = \frac{1}{Z}\sum_\delta e^{i k\cdot R_\delta}$$

$$E_k - E_0 \propto k^2$$

FM spin wave $\Delta E_k \propto k^2$



Particle number representation

High spin $\underline{S}$ spin → boson.

Holstein - Primakoff transformation

$$S_z = S - n$$

$$S^+|S_z\rangle = \sqrt{(S-S_z)(S+S_z+1)}\,|S_z+1\rangle$$

$$S^-|S_z\rangle = \sqrt{(S+S_z)(S-S_z+1)}\,|S_z-1\rangle$$

$$\Downarrow$$

spin guess Boson

$|S_z\rangle \Longrightarrow |n\rangle$

$|S_z\pm1\rangle \Longrightarrow |n\mp1\rangle$

$$\left\{\begin{aligned} S^+|n\rangle &= \sqrt{2S+1-n}\cdot \boxed{\overset{a|n\rangle}{\sqrt{n}\;|n-1\rangle}} \\ S^-|n\rangle &= \sqrt{2S-n}\cdot \boxed{\overset{a^+|n\rangle}{\sqrt{n+1}\;|n+1\rangle}} \\ S_z|n\rangle &= (\overset{S_z}{\overset{\shortparallel}{S-n}})\,|n\rangle \end{aligned}\right.$$

$n = S - S_z$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$$\left\{\begin{aligned} [a_k, a_{k'}^+] &= \delta_{kk'} \\ [a_k, a_{k'}] &= [a_k^+, a_{k'}^+] = 0 \end{aligned}\right.$$

Boson operator

large S

H - P :

$$\left\{\begin{aligned} S_m^+ &= \sqrt{2S - a_m^+ a_m}\cdot a_m \\ S_m^- &= a_m^+\sqrt{2S - a_m^+ a_m} \\ S_m^z &= S - a_m^+ a_m \end{aligned}\right. \simeq \left\{\begin{aligned} S_m^+ &= \sqrt{2S}\;a_m \quad \checkmark \\ S_m^- &= a_m^+\sqrt{2S} \\ S_m^z &= S - a_m^+ a_m \end{aligned}\right.$$

↓

Spin operator

↑ number

spin ½ $S = \frac{1}{2}$

$$\hat{H} = -J \sum_{m,\delta} \left[\hat{S}^z_m \hat{S}^z_{m+\delta} + \frac{1}{2}\left(\hat{S}^+_m \hat{S}^-_{m+\delta} + \hat{S}^-_m \hat{S}^+_{m+\delta} \right) \right]$$

leading order ⇕ high order

$$= E_0 + JS \sum_m \underbrace{(a^+_{m+\delta} - a^+_m)(a_{m+\delta} - a_m)} + \cdot\, O(S^0)$$

↓ from real space to k space.

$$= E_0 + 2JzS \sum_k (1-\gamma_k)\, a^+_k a_k$$

$$+ \frac{zJ}{2N} \sum_{k,k',q} \left(\gamma_{k-q} + \gamma_{k'} - 2\gamma_{k-q-k'} \right) a^+_{k-q}\, a^+_{k'+q}\, a_{k'}\, a_k$$

magnon-magnon interaction

FM: Ground state

excitation: spin wave;

magnon;

when $k \to 0$, we have $1-\gamma_k \longrightarrow k^2$.

$$E(k) - E_0 \propto k^2 \equiv Ak^2$$

magnetization: $M = N\cdot S - \sum_m a^+_m a_m$

$\langle S^z \rangle$

$$= N\cdot S - \sum_k a^+_k a_k = NS - \sum_k n(\varepsilon_k)$$

$$M = N\cdot S - \frac{V}{(2\pi)^d} \int \frac{C\cdot k^{d-1}\, dk}{e^{\frac{Ak^2}{k_BT}} - 1} \propto \frac{Ak^2}{k_BT}$$

$T > 0$ $\boxed{\text{when } k \to 0}$ $d \le 2$; integral $k \to 0$ divergence

$d \le 2$

FM; $T > 0$; No long-range order FM

* Hohenberg - Mermin - Wagner theorem *

$d=3$, $\quad M = NS - \dfrac{V}{(2\pi)^3}\displaystyle\int_0^\infty \frac{4\pi\, k^2\, dk}{e^{\frac{Ak^2}{k_B T}} - 1}$

$$M = NS\left[1 - \left(\frac{T}{T_0}\right)^{3/2}\right]; \quad \boxed{T_0 \sim J/k_B}$$

T_0 magnon BEC

FM spin wave

$\underline{\varepsilon_k \propto k^2}$

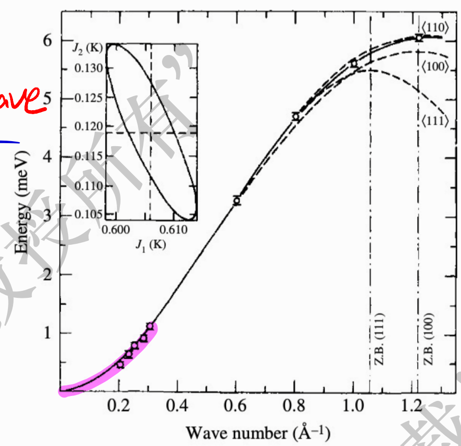


Figure 2.11 Spin-wave spectrum of europium oxide as measured by inelastic neutron scattering at a reference temperature of 5.5 K. Note that, at low values of momenta q, the dispersion is quadratic, in agreement with the low-energy theory. (Exercise: A closer inspection of the data shows the existence of a small gap in the spectrum at $\mathbf{q} = 0$. To what may this gap be attributed?) Figure reprinted with permission from L. Passell, O. W. Dietrich, and J. Als-Nielser, Neutron scattering from the Heisenberg ferromagnets EuO and EuS I: the exchange interaction, *Phys. Rev. B* **14** (1976), 4897–907. Copyright (1976) by the American Physical Society.

① Spin ½ HP fails

Spin $\longleftrightarrow$ Boson

② semi-classical argument

H-P 1940

Spin-½ Matsubara 1956. Prog. Theor. phys. 16 569

$S = 1/2$, $\quad S_j^{\pm} = S_j^x \pm i S_j^y \qquad S_j^z$

FM Ground state

commute relation

$$i \neq j \quad \underbrace{[S_i^+, S_j^+]_- = [S_i^-, S_j^-]_- = [S_i^-, S_j^+]_- = 0}$$

anti-Commute

$$\begin{cases} [S_i^+, S_i^+]_+ = [S_i^-, S_i^-]_+ = 0 & i = j. \\ \underbrace{[S_i^-, S_i^+]_+ = 1} & ; \quad S_i^z = \frac{1}{2} - S_i^+ S_i^- \end{cases}$$

$\underbrace{\text{Hard-core Bose}}$ $\leftarrow$ Huang K. S. Statistical Mechanics.

Commute $\quad [a_i^+, a_j^+]_- = [a_i, a_j]_- = [a_i, a_j^+]_- = 0 \quad i \neq j.$

anti-commute ← $\underbrace{[a_i, a_i^\dagger]_+ = 1; \quad [a_i^\dagger, a_i^\dagger]_+ = [a_i, a_i]_+ = 0}_{\{\ ,\ \}}; \quad i=j.$

Spin-1/2 mapping Hard-core bose.

$$\mathcal{H} = -J\sum_{\langle ij\rangle} \hat{S}_i \hat{S}_j$$

$$= J/2 \sum_{\langle i,j\rangle} \underbrace{(a_i^\dagger - a_j^\dagger)(a_i - a_j)} - J\sum_{\langle ij\rangle} \underbrace{a_i^\dagger a_i a_j a_j}.$$

$$= 2Jz\cdot\tfrac{1}{2}\sum_k (1-\gamma_k)\, \underline{a_k^\dagger a_k} + \cdots \qquad \text{interaction..}$$

☆ $\underbrace{\text{Spin-1/2 FM}}_{\text{magnon BEC}}$ mapping to $\underbrace{\text{hard core Bose.}}_{\text{weakly interacting BEC}}$

Semi-classical argument of FM spin wave.

Single site $\quad \mathcal{H} = -\lambda S^z$

$$\dot{S}^+(t) = \frac{i}{\hbar}[\mathcal{H}, S^+] = -i\underline{\lambda} S^+(t)$$

$$S^+(t) = e^{-i\lambda t} S^+(0)$$

$$\mathcal{H} = \hbar J \sum_\delta \hat{S}_i \hat{S}_{i+\delta}$$

$$\hat{S}_i\cdot\hat{S}_{i+\delta} = \hat{S}_i^z\, \underline{\underline{\hat{S}_{i+\delta}^z}} + \tfrac{1}{2}(\hat{S}_i^+ \hat{S}_{i+\delta}^- + \hat{S}_{i+\delta}^- \hat{S}_i^+)$$

☆ $$\underline{\underline{\dot{\hat{S}}_i^+}} = \frac{i}{\hbar}[H, \hat{S}_i^+]$$

$$= -i\cdot|J|\sum_\delta \left\{-\hat{S}_i^z\, \underline{\hat{S}_{i+\delta}^+} + \underline{\hat{S}_{i+\delta}^z}\, \underline{\underline{\hat{S}_i^+}}\right\}$$

$$\hat{S}_i^z = \hat{S}_{i+\delta}^z = \underline{\underline{\hbar S}} \qquad \underline{\underline{S \gg 1}} \qquad \underline{\underline{\text{large spin limit}}}$$