REFERENCES

1. Miller, S. E., "Integrated Optics: An introduction," this issue, pp. 2059–2069.
2. Kapany, N. S., *Fiber Optics,* New York: Academic Press, 1967, pp. 36–80.
3. Bracey, M. F., Cullen, A. L., Gillespie, E. F. F., and Staniforth, J. A., "Surface Wave Research in Sheffield," I.R.E. Trans. Antennas and Propagation, *AP-7,* No. 10 (December 1959), pp. 219–225.
4. Snitzer, E., "Cylindrical Dielectric Waveguide Modes," J. Opt. Soc. of Amer., *51,* No. 5 (May 1961), pp. 491–498.
5. Snitzer, E., and Osterberg, H., "Observed Dielectric Waveguide Modes in the Visible Spectrum," J. Opt. Soc. of Amer., *51,* No. 5 (May 1961), pp. 491–505.
6. Marcatili, E. A. J., "Dielectric Rectangular Waveguide and Directional Coupler for Integrated Optics," this issue, pp. 2071–2102.
7. Schlosser, W., and Unger, H. G., "Partially Filled Waveguides and Surface Waveguides of Rectangular Cross-Section," *Advances in Microwaves,* New York: Academic Press, 1966, pp. 319–387.
8. Shaw, C. B., French, B. T., and Warner, C. III, "Further Research on Optical Transmission Lines," Sci. Rep. No. 2, Contract AF449 (638)-1504 AD 625 501, Autonetics Report No. C7-929/501, pp. 13–44.
9. Stratton, J. A., "Electromagnetic Theory," New York: McGraw-Hill, 1941, p. 361.
10. Laxpati, S. R., and Mittra, R., "Energy Considerations in Open and Closed Waveguides," IEEE Trans. Antennas and Propagation, *AP-13,* No. 6 (November 1965), pp. 883–890.
11. Hamming, R. W., *Numerical Analysis for Scientists and Engineers,* New York: McGraw-Hill, 1962, pp. 81–82.
12. Freed, B. H., "Algorithm 41," Revision Evaluation of Determinant Comm. ACM, *6,* No. 9 (September 1963), p. 520.
13. "System/360 Scientific Subroutine Package," IBM, White Plains, N. Y., H20-0205-2, pp. 179–182.
14. Collin, R. E., "Field Theory of Guided Waves," New York: McGraw-Hill, 1960, pp. 480–495.

Improved Relations Describing Directional Control in Electromagnetic Wave Guidance

By E. A. J. MARCATILI and S. E. MILLER

(Manuscript received January 22, 1969)

The direction-changing capability of electromagnetic waveguides may be limited not only by mode conversion but also by radiation if the transverse field extends indefinitely into a freely propagating region. This paper gives new, more accurate expressions for the permitted bending radius with respect to mode conversion, using coupled-wave theory to categorize the wide variety of transmission media possible. This paper also makes a suggestion for estimating the permitted bending radius when radiation is a limitation. In single-mode "open" waveguides that have transverse fields extending indefinitely into a freely propagating region (such as a dielectric waveguide), the permitted bending radius is limited by radiation effects, whereas in either the open or completely shielded multimode waveguides, the permitted bending radius is usually limited by mode conversion.

I. INTRODUCTION

It is useful to be able to characterize the direction-changing capability of electromagnetic waveguides without detailed knowledge of the waveguiding structure. The first work in this area was reported by Miller in 1964.[1] A direction-determining parameter $R_{\min}$ was defined

$$R_{\min} = \frac{a^3}{4\lambda^2} \tag{1}$$

in which $R_{\min}$ is a bend radius, a is the full transverse width of the field distribution, and λ is the wavelength in the medium in which the waveguide is embedded.* For bend radii longer than $R_{\min}$, Ref. 1 indicates that wave propagation is virtually as in a straight guide; at radii less than $R_{\min}$ something drastic happens. Just what changes

* Notice that we have redefined a here; in Ref. 1 the full transverse width of the field distribution was $2a$.

occur in a straight guide depends on the nature of the medium in detail; for hollow conducting guides the change is large mode conversion and for beam transmission in a sequence of infinitely wide lenses the change is also mode conversion appearing as a wide oscillation of the beam about the nominal axis of propagation.

Following similar lines of thought, a parameter

$$\delta_{max} = \frac{\lambda}{a} \tag{2}$$

is given to describe the transition region between essentially normal wave propagation and the region of drastic changes for abrupt angular changes in direction.[1] The only restriction on these order of magnitude direction-determining parameters given in Ref. 1 is the exclusion of degeneracy between the used mode and some other mode coupled by the direction change. It is well known that such a degeneracy results in complete loss of signal for certain lengths of bent guide regardless of the bending radius, and that removal of the degeneracy by dissipative or reactive means can in principle make the bend loss as small as desired.[2–4]

In recent studies of bend losses in dielectric waveguides, Marcatili found a serious disagreement between the implications of equation (1) and the bend losses predicted by analysis of the particular wave-guiding structure.[5] For an "open" waveguide—that is, one in which the transverse field decays exponentially in a transverse plane but extends to great distances—he found that the bend radius required for tolerable losses was much larger than given by equation (1) and it followed a different law with relation to a and λ when only one mode could propagate.

It is now clear that two components of bend loss must be considered: the dissipative loss (resulting from either radiation or coupling to a high-loss undesired mode) for the normal mode of the bend region characterized by an attenuation coefficient α_r, and the mode conversion loss P_c for the straight-guide mode on entering and leaving the curved region. If mode transformers were used at the ends of the curved region (impractical for occasional bends in most transmission situations), the mode conversion loss would be zero and any bend R would be acceptable from that criterion.

Equation (1) relates to the mode conversion loss; it fails to give a correct estimate when dissipative loss is important. The permitted bend radius R must be assessed with respect to dissipative loss as

well as mode conversion loss; Section II gives relations which make this possible. Improved forms of equations (1) and (2) have also been derived which explicitly relate the maximum conversion loss to the bending radius for the generalized electromagnetic waveguide. The added quantitative factor should provide greater usefulness since the improved relations not only identify the transition region between virtually straight-guide behavior and violent changes, but also give detail about the transition. Section III gives these results and the appendices give the derivations.

II. RADIATION FROM CURVED OPEN WAVEGUIDES

Figure 1 shows a representation of an open waveguide. The shaded wave-guiding region has an effective index of refraction larger than that of the surrounding region, resulting in a transverse field distribution for the guided mode $F(x)$ which decays exponentially but remains finite. To derive a generalized expression for radiation loss as a function of bending radius R, we visualize this as a two-dimensional guide with an isotropic surrounding region capable of supporting a free-space radiating wave. We note that at some transverse distance x_r the maintenance of a pure guided mode with equiphase fronts on

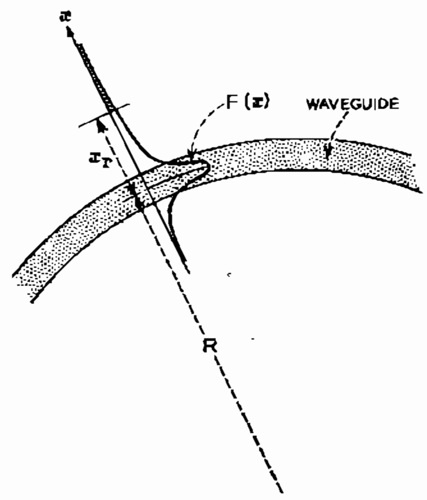

Fig. 1 — A two-dimensional open waveguide.

radial planes requires energy propagating at the speed of light, and for $x > x_r$ a pure guided mode implies energy propagating at greater than the velocity of light. This is true at some value of x_r for any finite bend radius R, since $F(x)$ extends indefinitely in the x direction. We postulate that the transverse field distribution $F(x)$ is virtually the same in the curved region as in a straight guide for large R. The fraction of the energy in the guided mode at $x > x_r$ is assumed to be lost to radiation; this loss is taken to occur in a longitudinal distance equal to the collimated-beam length associated with the field $F(x)$. All these assumptions imply that any mode propagating along the curved open guide radiates. This is indeed the case for the modes in the curved dielectric guide analyzed in Ref. 5.

As developed in Appendix A, the attenuation coefficient for the normal mode of the bend region is

$$\alpha_r = \frac{1}{2Z_c}\frac{\mathcal{E}_l}{\mathcal{E}_T}, \tag{3}$$

where

$$\mathcal{E}_l = \int_{x_r}^{\infty} F^2(x)\,dx, \tag{4}$$

$$\mathcal{E}_T = \int_{-\infty}^{\infty} F^2(x)\,dx, \tag{5}$$

$$Z_c = \frac{a^2}{2\lambda_s}, \tag{6}$$

$$x_r = \frac{(k_z - k_s)}{k_s}R, \tag{7}$$

k_z = longitudinal phase constant for the guided mode,
$k_s = 2\pi/\lambda_s$ phase constant for a plane wave in the surrounding region, and
a = effective width of the transverse field $F(x)$.

Applying this formulation to a curved two-dimensional dielectric-slab waveguide of width t gives the following. From solutions of Maxwell's equations in a straight guide

$$F(x) = \cos k_x x \quad \text{for} \quad -\frac{t}{2} \leqq x \leqq \frac{t}{2}, \tag{8}$$

$$F(x) = \cos\left(\frac{k_x t}{2}\right) e^{\frac{-\left(|x| - \frac{t}{2}\right)}{\xi}} \quad \text{for} \quad |x| \geqq \frac{t}{2}. \tag{9}$$

The resulting expressions for z_c, $\mathcal{E}_l$, and $\mathcal{E}_T$ are

$$\mathcal{E}_l = \frac{\xi}{2}\cos^2\left(\frac{k_x t}{2}\right) e^{\frac{-2\left(x_r - \frac{t}{2}\right)}{\xi}}, \tag{10}$$

$$\mathcal{E}_T = \frac{t}{2} + \frac{1}{2k_x}\sin k_x t + \xi\cos^2\left(\frac{k_x t}{2}\right), \tag{11}$$

$$z_c = \frac{\left[t + 2\xi\cos\left(\frac{k_x t}{2}\right)\right]^2}{2\lambda_s}. \tag{12}$$

These expressions, when put into equation (3), yield a radiation attenuation coefficient of the form*

$$\alpha_r = c_1 \exp(-c_2 R), \tag{13}$$

where c_1 and c_2 are independent of R. As Table I illustrates, in several cases of interest c_1 and c_2 are very large numbers (calculated for $\lambda = 0.6328$ μm). Case 1 corresponds to a thin glass sheet surrounded by air; cases 2 and 3 correspond to 1 percent and 0.1 percent index differences between the guide and the surrounding region, a possible guide of interest for miniature laser-beam circuitry.[6] Because c_1 and c_2 are so large, reasonable values of α_r occur only within a narrow range of bend radius R. Figure 2 illustrates α_r versus R for case 2. We can define a transition radius R_t as that value of R which gives α_r = 1 neper per meter:

$$R_t = \frac{1}{c_2}\log c_1 \tag{14}$$

in which c_1 and c_2 are the constants of equation (13) found by evaluating equation (3). Because of the exponential nature of α_r versus R, radii smaller than R_t give excessive losses and radii slightly larger than R_t give negligibly small losses. We may therefore use R_t as an index of this transition for radiation losses analogous to the R_{min} of equation (1) for mode conversion losses.

Notice the size of x_r, the transverse distance to where wave propagation at the velocity of light is required. For cases 1, 2, and 3, x_r has the values 1.0, 3.9, and 16.5 μm, respectively, for α_r = 1 neper per meter. Wave propagation at the velocity of light occurs quite close to the center of the guide, well within the bending radius.

* This paper uses mks units in all formulas.

TABLE I—VALUES FOR c_1 AND c_2

Case	Waveguide index of refraction	Slab width t (μm)	Surrounding index of refraction	c_1 (nepers per meter)	c_2 (meters^{-1})	R for $\alpha_r = 1$ neper/m
1	1.5	0.198	1.0	2.57×10^6	3.47×10^6	4.25 μm
2	1.5	1.04	1.485	1.04×10^5	1.46×10^4	0.79 mm
3	1.5	1.18	1.4985	5.4×10^3	81.4	0.106 m

In Appendix A the results using equation (3) are compared with the more exact values of α_r obtained from Maxwell's equations directly.[5] For a given α_r equation (3) yields a value of R about 0.6 times that obtained from Ref. 5. Moreover, Ref. 5 shows that, as the slab width t increases, the radiation loss does not decline indefinitely; the normal mode transverse field reshapes itself in the bend to increase $F(x)$ in the x_r region. However, the mode conversion loss usually becomes important at those values of t and for incidental bends (that is, without mode matching transformers) the mode conversion loss is limiting rather than radiation loss.

Another approach, which yields an expression for the radiation loss of the curved guide in terms of constants of the straight guide, consists

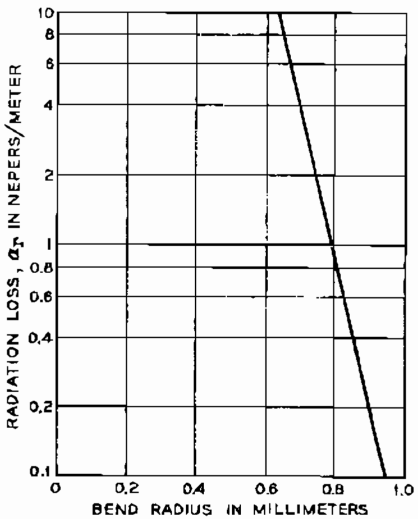

Fig. 2—Radiation loss versus bend radius for a two-dimensional dielectric waveguide; case 2 of Table I.

of noticing that the boundary value problem, which can be solved exactly by matching the radial impedances at each interface, can also be solved approximately if the radius of curvature R is so large that the field components of the curved guide differ only slightly from those in the straight guide.[5] Then, all the impedances can be replaced by those of the straight guide except that on the external interface of the bend which, according to Ref. 5, must be multipled by

$$1 + i \exp\left(-\tfrac{2}{3}R\,\frac{k_{xe}^3}{k_z^2}\right).$$

In this expression k_{xe} and k_z are the propagation constants in the x and z directions in the external medium of the straight guide. The attenuation constant of the curved guide results

$$\alpha_r = k_{xe} \exp\left(-\tfrac{2}{3}R\,\frac{k_{xe}^3}{k_z^2}\right)\frac{\partial k_z}{\partial k_{xe}}. \tag{15}$$

This expression should give greater accuracy in general and does so in the case of the slab waveguide used in this section. It also shows that waveguides which present imaginary radial impedances have no radiation loss.

III. MODE CONVERSION LOSSES IN CURVED OPEN OR BOUNDED WAVEGUIDES

3.1 *General Formulation of Tilt Relation*

When a pure mode of a straight multimode waveguide enters and leaves a curved region, it generally suffers mode conversion loss. Coupled-mode theory has been applied to calculate these losses as a function of bend radius and to devise lower loss bend structures.[3,4,7,8] In these previous contributions, direct solution of Maxwell's equations is used to find which of the straight-guide modes are coupled in the bend, and for these important modes to find the transfer coupling coefficients and the associated differences in propagation constants which are needed in the coupled wave solution.

We present here a generalized use of coupled wave theory which gives an improvement on equations (1) and (2) in predicting approximate values of tolerable bend radius without direct solution for the transfer coupling coefficients or the phase constants. We do not imply that this provides accuracy comparable to a direct solution. It does yield an approximate answer to show where further work to get more accuracy is of interest.

The first approximation is used to derive the transfer coupling coefficient from the self-coupling coefficient. Consider a tilt (illustrated in Fig. 3) for a hollow metallic rectangular waveguide. The self-coupling in the tilt from the incident mode to the same mode beyond the tilt, of angle δ, is[9]

$$|c_{ss}| = \left| \frac{\int_0^w \int_0^b \left[\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2\right] \exp\left(i \frac{2\pi\,\delta}{\lambda_s} x\right) dx\,dy}{\int_0^w \int_0^b \left[\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2\right] dx\,dy} \right|, \tag{16}$$

in which λ_s is the guided wavelength along z.

The function F is the axial field component which, for hollow metallic rectangular waveguides, is either $\sin \pi p\, x/w \sin \pi q\, y/b$ for TM_{pq} modes or $\cos \pi p x/w \cos \pi q\, y/b$ for TE_{pq} modes.

For small tilt angles $|c_{ss}|$ is of the form

$$|c_{ss}| = 1 - \Delta, \tag{17}$$

where $\Delta \ll 1$; Δ corresponds to the energy lost from the input mode at the tilt, whether by reflection or transmission into a single or into many modes. We now assume the incident mode to be well above cutoff so that reflection effects are small; that is, $w/\lambda > 1$ and preferably $w/\lambda \gg 1$. We further assume that all the lost energy at the tilt goes into a single undesired mode. For such a transfer

$$|c_{ss}| = (1 - |c_t^2|)^{\frac{1}{2}} \approx 1 - \tfrac{1}{2}|c_t|^2, \tag{18}$$

where c_t is the transfer coupling coefficient. We then combine equations (17) and (18) to obtain the transfer coupling coefficient

$$|c_t| = (2\Delta)^{\frac{1}{2}}, \tag{19}$$

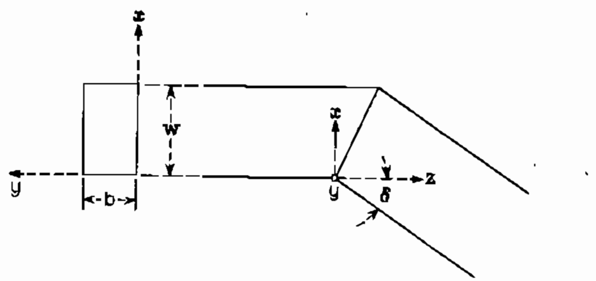

Fig. 3 — Tilt in hollow metallic rectangular waveguide.

and the fraction of the input power that is converted is

$$P_t = 2\Delta. \tag{20}$$

Carrying out the integration of equation (16) for the rectangular metallic waveguide, assuming $\delta w/\lambda_s \ll 1$, gives

$$P_t = B\left(\frac{\delta w}{\lambda_s}\right)^2. \tag{21}$$

Appendix C shows that for the lowest order TE mode TE_{10}, B is 5.28. For other modes, B ranges between 5.28 and 1.28; we somewhat arbitrarily select the geometric mean of these values to approximate P_t for any mode. Then,

$$P_t = 2.6\left(\frac{\delta w}{\lambda_s}\right)^2, \tag{22}$$

$$c_t = 1.61\left(\frac{\delta w}{\lambda_s}\right), \tag{23}$$

$$\delta = 0.62 \frac{\lambda_s}{w} (P_t)^{\frac{1}{2}}, \tag{24}$$

which we have derived under the restrictions

$$\frac{w}{\lambda} \gg 1, \qquad \frac{\delta w}{\lambda} \ll 1.$$

Equation (24) is an improved form of equation (2). It shows the approximate tilt angle permitted versus fractional power converted. Derived for hollow metallic waveguide of width w, the "field" width is also w which is equivalent to a in equation (2); since we required the modes to be far from cutoff, $\lambda_s \cong \lambda$; however, we note that the converted power P_t is smaller in fact than indicated by using $\lambda_s = \lambda$ since the guided wavelength λ_s is greater than λ.

3.2 *Formulation of Bend Coupling Coefficient*

Using a limiting process, described in Section 2.3.2 of Ref. 10, the tilt conversion coefficient can be converted to a continuous bend conversion coefficient. Consider a sequence of straight guide sections, each of length l and connected making a tilt angle δ (Fig. 4). Let us assume that a mode entering in this guide couples at each tilt mostly to itself and lightly to one single spurious mode travelling in the forward direction. The tilt amplitude coupling coefficient is given by equation (23). The coupling per unit length is $|c_t/l|$; letting l and δ

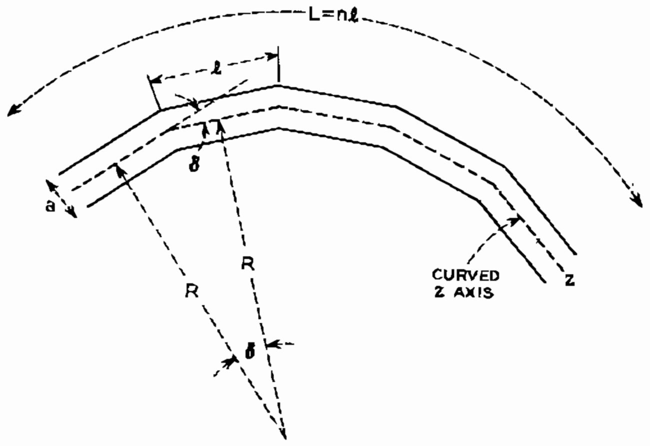

Fig. 4 — Waveguide bend made of a series of straight segments.

go to zero simultaneously in such a way that $l/\delta = R$, the bend amplitude coupling coefficient c_B is:

$$|c_B| = 1.61 \frac{w}{\lambda_s R}. \quad (25)$$

3.3 *Coupled Wave Interaction*

We are now prepared to discuss the effect of bends in producing mode conversion using coupled-wave theory. In this approach the signal amplitude E_1 is related to the undesired mode amplitude E_2 by the equations

$$\frac{dE_1}{dz} = -\Gamma_1 E_1 + kE_2 , \quad (26)$$

$$\frac{dE_2}{dz} = -\Gamma_2 E_2 + kE_1 , \quad (27)$$

in which

$\Gamma_1 = \alpha_1 + i\beta_1$ = propagation constant of signal wave,
$\Gamma_2 = \alpha_2 + i\beta_2$ = propagation constant of undesired wave, and
k = transfer coupling coefficient.

These equations have been solved and the resulting wave interactions discussed in many papers.[3,4,8,10,11] Appendix B gives a few of the expressions relevant to this disscussion; we will draw from these. We assume a boundary condition, $E_1 = 1.0$ and $E_2 = 0$ at $z = 0$, throughout. The effects of mode coupling depend importantly on $(\Gamma_1 - \Gamma_2)$ and k. In finding expressions which improve on equation (1) we break the discussion of a generalized waveguide down into a series of cases which are classified by the relation between the coupling coefficient k and $(\Gamma_1 - \Gamma_2)$.

3.4 *Gradual Bends in Low-Loss Waveguides*

We categorize the case of gradual bends in low-loss waveguides by

$$|k^2| \ll (\beta_1 - \beta_2)^2, \quad (28)$$

$$(\alpha_1 - \alpha_2)^2 \ll (\beta_1 - \beta_2)^2, \quad (29)$$

$$\alpha_2 L \ll 1, \quad (30)$$

where L is the length of the bend.

This is the most likely case to be encountered in waveguides intended for low-loss transmission. The special case of degeneracy, $\beta_1 = \beta_2$, is treated in Section 3.6; degeneracy is not likely to occur accidentally since it is a very critical condition. Because β is very large compared with α in typical cases, equation (29) can be satisfied with relatively small changes from the degenerate condition, and the present case can be considered achievable except under very special circumstances.

With small α's, k is pure imaginary, $k = ic$; a value such as given by equation (25) applies. With equation (30) valid, the signal loss oscillates along the bend between zero and a maximum value

$$P_c = \left(\frac{2c}{\beta_1 - \beta_2}\right)^2. \quad (31)$$

To complete our derivation we need $(\beta_1 - \beta_2)$, which should be the difference between the phase constants of the modes coupled in the bend. We have not determined in our generalized waveguide case just which modes are coupled. We use as an approximation the rectangular metallic waveguide case of Fig. 3, and calculate the $\Delta\beta$ for the pth and $(p \pm 1)$ mode; again requiring the modes to be far from cutoff, we find

$$\Delta\beta = (\beta_1 - \beta_2) \simeq \frac{(2p \pm 1)\pi}{4} \frac{\lambda}{w^2}. \quad (32)$$

Combining equations (31), (32), and (25) with $c = |c_B|$ and solving for R yields

$$R = \frac{4.1}{(2p \pm 1)(P_c)^{\frac{1}{2}}} \frac{w^3}{\lambda^2}. \tag{33}$$

For the $p = 1$ mode only, the $(+)$ sign in $(2p \pm 1)$ applies; but for higher order modes either sign is applicable and the $(-)$ sign will be controlling. As a further rough approximation we may drop the ± 1 term, yielding

$$R = \frac{2.05}{p(P_c)^{\frac{1}{2}}} \frac{w^3}{\lambda^2}. \tag{34}$$

Equation (34) has the same general form as equation (1) but gives added accuracy by showing the quantitative influence of mode index and fractional conversion loss permitted.

3.5 *Gradual Bends in Lossy Waveguides*

Here we keep equations (28) and (29) but address the case where the undesired mode coupled to has high loss over the length L of the bend:

$$\alpha_2 L \gg 1. \tag{35}$$

Now, the true situation is very complex. The coupling coefficient k is complex and may have real and imaginary components that are equal. Energy conservation between $c_{\circ}$ and c_i, which was implied by equation (18), is not justified. Experience with helix waveguide for TE^0_{01} waves shows, however, that the modulus of the helix coupling coefficient is comparable to that for a copper tube; therefore, we use equation (25) for the $|k|$ and proceed as before.

As the result of equation (35) the oscillations in the conversion loss are damped out and the conversion loss has the form of a simple exponential; that is, the normal mode of the curved region is set up with an attenuation coefficient $(\alpha_B + \alpha_1)$, where the extra loss resulting from the bend is

$$\alpha_B = \text{real}\left[\frac{k^2}{(\Gamma_1 - \Gamma_2)}\right]. \tag{36}$$

Using equation (25) with $|c_B| = |k|$, this becomes

$$\alpha_B = \frac{4.21}{(2p \pm 1)^2} \frac{(\alpha_2 - \alpha_1)w^6}{R^2\lambda^4}. \tag{37}$$

This resembles a radiation loss in that it grows with length L, whereas in Section 3.4 the oscillatory loss peak was independent of L.

We can rearrange equation (37) to show the permitted bend radius R, (again dropping the ± 1):

$$R = \frac{1.05}{p}\left[\frac{(\alpha_2 - \alpha_1)}{\alpha_B}\right]^{\frac{1}{2}} \frac{w^3}{\lambda^2}. \tag{38}$$

Here α_B may be regarded as a design criterion selected to meet the requirements of a particular use, analogous to P_c above; as such α_B may be independent of λ or may have some λ dependency.

Expression (38) has a character markedly different from equation (1). Since α_2 and α_1 are dependent on guide size and wavelength the a^3/λ^2 dependence given by equation (1) is not valid when coupling takes place to a very lossy mode.

3.6 *Bends in a Waveguide with Low-Loss Degenerate Coupled Modes*

When the modes coupled in the bend are degenerate, whether by design or misfortune, a far more stringent requirement on R develops. In this case

$$\beta_1 \doteq \beta_2 . \tag{39}$$

Because attenuation coefficients are small in many typical cases, it is relatively easy to obtain coupling coefficients that are larger, that is,

$$|c_B|^2 \gg |\alpha_2 - \alpha_1|^2. \tag{40}$$

Then the signal wave output of a bend of length L is

$$|E_1| = |\cos c_B L| \tag{41}$$

or, using the value of equation (25) for c_B,

$$|E_1| = \left|\cos\left(\frac{1.61\, wL}{\lambda_s R}\right)\right|. \tag{42}$$

The signal loss is infinite when the argument of the cosine is an odd multiple of $\pi/2$, and the corresponding bend radius R_∞ or bend length L_∞ are

$$R_\infty = \frac{1.02\, wL}{m\lambda_s} \tag{43}$$

$$L_\infty = 0.98\, m \frac{\lambda_s R}{w} \tag{44}$$

for $m = 1, 3, 5.$

For small fractional power losses P_c, equation (42) may be approximated by the first term of the expansion; the resulting permitted bend radius is

$$R = \frac{1.61}{(P_e)^{\frac{1}{2}}} \frac{wL}{\lambda}. \tag{45}$$

When $(\beta_1 - \beta_2)$ is nonzero, the signal transmission oscillates between unity and a minimum of

$$|E_1|_{\min} = \frac{\left|\dfrac{\beta_1 - \beta_2}{2c}\right|}{\left[\left(\dfrac{\beta_1 - \beta_2}{2c}\right)^2 + 1\right]^{\frac{1}{2}}} \tag{46}$$

which merges with equation (30) and the case considered in Section 3.4.

3.7 *Bends in Waveguides with High-Loss Degenerate Coupled Modes*

When the phase constants of the modes coupled in the bend are degenerate—that is, equation (37) holds—but the undesired mode is very lossy

$$|\alpha_2 - \alpha_1|^2 \gg |c_B|^2. \tag{47}$$

Then Appendix B shows that we again have normal-mode propagation in the bend region (as in Section 3.5) with an attenuation constant $(\alpha_1 + \alpha_B)$ where

$$\alpha_B = \frac{c_B^2}{\alpha_2 - \alpha_1}. \tag{48}$$

Using equation (25), this yields a bend radius:

$$R = \frac{1.61}{[\alpha_B(\alpha_2 - \alpha_1)]^{\frac{1}{2}}} \frac{w}{\lambda}. \tag{49}$$

This corresponds to very long bend radii in order to have equation (47) valid. Just as in equation (38), α_B of equation (49) is a discretionary design parameter.

IV. COMPARISON WITH KNOWN DIRECT SOLUTIONS

The principal usefulness of the preceding approximate relations for permissible tilt and bend radius is in new unstudied situations, where direct solutions are not available. However, we compare here the approximations with known direct solutions in order to gauge the accuracy to be expected.

4.1 *Tilt in a Sequence of Cylindrical Lenses: (Two-dimensional Problem)*

The input mode is gaussian, its spot size is w_o, and the transverse field distribution is $\exp[-(x/w_o)^2]$. The normalized power coupled to other modes at the tilt ($\delta \ll 1$) is[12]

$$P_2 = 1 - \left\{\frac{\int_{-\infty}^{\infty} \exp\left[-2\left(\frac{x}{w_o}\right)^2 - i\frac{2\pi}{\lambda}\delta x\right] dx}{\int_{-\infty}^{\infty} \exp\left[-2\left(\frac{x}{w_o}\right)^2\right] dx}\right\} \cong \left(\frac{\pi\,\delta w_o}{\lambda}\right)^2. \tag{50}$$

To compare this exact result with our approximate one, equation (22), we must define the width a of the beam. Somewhat arbitrarily we choose

$$a = 2w_o; \tag{51}$$

thus 95 percent of the power is traveling within the width a.

Substituting this value in equation (50) we obtain

$$P_2 = 2.5\left(\frac{\delta a}{\lambda}\right)^2. \tag{52}$$

This compares to equation (21) with $p = 1$ and $w = a$,

$$P_t = 2.6\left(\frac{\delta a}{\lambda}\right)^2. \tag{53}$$

Considering that equation (53) came from rectangular metallic waveguide and equation (52) from an open lens waveguide, the correspondence seems excellent.

4.2 *Tilt in a Cylindrical Metallic Waveguide Propagating* TE_{01}^0

For TE_{01}^0 at a tilt, important coupling is known to occur to three modes:[2,10]

Mode pair	Tilt coupling coefficient	
$TE_{01}^0 - TE_{11}^0$	$0.585\,\frac{a\delta}{\lambda}$	(54)
$TE_{01}^0 - TE_{12}^0$	$0.98\,\frac{a\delta}{\lambda}$	(55)
$TE_{01}^0 - TM_{11}$	$0.58\,\frac{a\delta}{\lambda}$	(56)

where a is the diameter of the round guide and is the full width of the transverse field. This corresponds to equation (23) with $w = a$ and $p = 2$ (two extrema in the transverse field),

$$c_t = 1.61 \frac{a\delta}{\lambda}. \tag{57}$$

In the real case, the converted power is the sum of three conversions using the above three coupling coefficients; since the three components vary with a different period versus λ, or distance along the guide after the tilt, the actual mode conversion is a complicated function. We might take the root-sum-square combination of equations (54) through (56) to compare with equation (57), leading to

$$\mathrm{TE}_{01}^{0}\ c_{t(rss)} \cong 1.65 \frac{a\delta}{\lambda}. \tag{58}$$

The converted power loss is $|c_t|^2$, so we see that equation (57) gives a correct order of magnitude indication, but it lacks significant detail.

4.3 *Bends in Cylindrical Metallic Waveguide Propagating TE_{01}^{0}*

The above discussion for tilt coupling coefficient applies directly to bend coupling coefficient in empty round guides, noting the interrelation

$$|c_B| = \frac{|c_t|}{R\delta}. \tag{59}$$

However, the maximum conversion loss in the bend is also controlled by the quantity $(\beta_1 - \beta_2)$ as given in equation (31). For the three important modes, the values are

Mode	$\lvert \beta_1 - \beta_2 \rvert$	
$\mathrm{TE}_{01}^{0} - \mathrm{TE}_{11}^{0}$	$3.6 \frac{\lambda}{a^2}$	(60)
$\mathrm{TE}_{01}^{0} - \mathrm{TE}_{12}^{0}$	$4.4 \frac{\lambda}{a^2}$	(61)
$\mathrm{TE}_{01}^{0} - \mathrm{TM}_{11}^{0}$	0	(62)

where a is again the guide diameter. These are to be compared with equation (32) with $w = a$ and $p = 2$,

$$|\beta_1 - \beta_2| = 3.9 \frac{\lambda}{a^2}. \tag{63}$$

The approximation (63) agrees well with the values for the $\mathrm{TE}_{01}^{0} - \mathrm{TE}_{11}^{0}$ and $\mathrm{TE}_{01}^{0} - \mathrm{TE}_{12}^{0}$ from expressions (60) and (61). However expression (62) shows that empty round guide has a degeneracy, which controls its behavior.[2] The permitted bend radius is controlled by the $\mathrm{TE}_{01}^{0} - \mathrm{TM}_{11}^{0}$ interaction. Exact theory shows the bend length to the first extinction of signal is[2]

$$L_\infty = 2.7 \frac{R\lambda}{a}, \tag{64}$$

which is to be compared with equation (44) with $w = a$ and $m = 1$,

$$L_\infty = 0.98 \frac{R\lambda}{a}. \tag{65}$$

Here the agreement is again quite good. The permitted bend radius for P_c fractional power loss, from exact theory is

$$R = \frac{0.58}{(P_c)^{\frac{1}{2}}} \frac{aL}{\lambda}, \tag{66}$$

and the approximation from equation (45) is

$$R \doteq \frac{1.61}{(P_c)^{\frac{1}{2}}} \frac{aL}{\lambda}. \tag{67}$$

In practical use of round guides for TE_{01}, however, the bare pipe is modified to eliminate the degeneracy. Intentionally making the empty guide elliptical is one way;[3] it takes only 1.7 percent diameter difference to make $(\beta_1 - \beta_2)^2 = 10(\alpha_2 - \alpha_1)^2$, making the relations of Section 2.4 valid. A more symmetrical modification is to add a thin dielectric lining; with a polyethelyne lining only 0.010 inches thick in a 2 inch inner diameter guide, the $(\beta_1 - \beta_2)$ for $\mathrm{TE}_{01}^{0} - \mathrm{TM}_{11}^{0}$ is about 60 percent of that given above for $\mathrm{TE}_{01}^{0} - \mathrm{TE}_{12}^{0}$.[12] This also yields $(\beta_1 - \beta_2)^2 \gg (\alpha_2 - \alpha_1)^2$ for all modes. Interestingly, exact theory shows that the lining drops the $\mathrm{TE}_{01}^{0} - \mathrm{TE}_{11}^{0}$ bend coupling coefficient by an order of magnitude.[12,13] Thus only two small mode conversions occur in the bend of lined waveguide. Taking the simple sum of these conversion losses yields, from this "exact" treatment,

$$P_c = 0.098 \frac{a^6}{R^2\lambda^4}. \tag{68}$$

The exact radius relation is then

$$R = \frac{0.31}{(P_c)^{\frac{1}{2}}} \frac{a^3}{\lambda^2}. \tag{69}$$

This is to be compared with equation (33) with $w = a$ and $p = 2$,

$$R = \frac{1.02}{(P_c)^{1/2}} \frac{a^3}{\lambda^3}. \tag{70}$$

Considering the complexity of the true situation the estimate provided by equation (70) is good.

4.4 *Helix Waveguide for TE_{01}°*

The helix waveguide for TE_{01}° is a very special structure designed to maximize the attenuation to the undesired modes.[14,15] This waveguide is unusual in presenting very large $(\alpha_2 - \alpha_1)$. The bend coupling coefficients k of equations (26) and (27) are no longer pure imaginary as they were in the simple metallic tube. For example, the complex nature of the helix coupling coefficients are shown for comparison with those of a metallic tube; we set $k = c' + jc''$, as shown in Table II. The helix values correspond to a longitudinal wall impedance of 196 ohms with a capacitive angle of 5°, both guides at $\lambda = 5.4$ mm and a guide diameter of 5.08 cm.

The attenuation coefficient of the normal mode of the bend region is

$$\alpha_1 + \sum^{n} \mathrm{Real}\left[\frac{k_n^2}{(\Gamma_1 - \Gamma_n)}\right] \tag{71}$$

where the summation represents the contributions of the three modes above. Using the helix waveguide coupling values of Table II, the conversion loss contributions are given in Table III. Note that the contributions of the TE_{12} and TM_{11} modes are of opposite sign; experiment agrees well with this theory.[16] An approximate degeneracy exists between TM_{11} and TE_{12} in the helix waveguide.

When such direct computations were made over a range of numerical conditions in the 30 to 100 GHz region on helix waveguides varying in diameter from 0.25 inch to 3 inches, it was found that the mode conversion contribution to the bend-region normal-mode attenuation coefficient is approximately

$$\alpha_B = 0.009 \frac{a^3}{R^2 \lambda^{2.7}}, \tag{72}$$

which yields a permitted bend relation from direct solution of the helix problem:

$$R = \frac{0.095}{(\alpha_B)^{1/2}} \frac{a^{1.5}}{\lambda^{1.35}}. \tag{73}$$

The corresponding approximate relation from Section 3.5 is equation (38) with $w = a$ and $p = 2$,

$$R = 0.52\left(\frac{\alpha_2 - \alpha_1}{\alpha_B}\right)^{1/2} \frac{a^3}{\lambda^2}. \tag{74}$$

To compare functional dependence on a and λ, we need to know how $(\alpha_2 - \alpha_1)^{1/2}$ [which is $(\alpha_2)^{1/2}$] varies with a and λ in the helix waveguide. Unfortunately this is not readily available although it was implicitly used in the work which yielded equation (72). However, a single numerical point is known: at $a = 5.08$ cm and $\lambda = 5.4$ mm, $\alpha_2 = 1.4$ nepers per meter for TM_{11}, which will control the guide behavior in equation (74). With these numbers equation (73) yields

$$R_{\mathrm{exact}} \cong \frac{1.12}{(\alpha_B)^{1/2}} \tag{75}$$

whereas equation (74) yields

$$R_{\mathrm{approx}} = \frac{2.76}{(\alpha_B)^{1/2}}. \tag{76}$$

The approximation is only off a factor of about two, which is remarkable and may be fortuitous. We suggest that equations (38) and (74) be considered provisional until proven or disproven by additional work.

TABLE II—HELIX WAVEGUIDE COUPLING VALUES

Mode	Solid Metallic Tube		Helix Waveguide	
	$c'R$	$c''R$	$c'R$	$c''R$
TE_{11}	0	5.5	−0.16	6.86
TM_{11}	0	5.46	−8.03	−5.71
TE_{12}	0	9.21	−3.76	11.88

TABLE III—CONVERSION LOSS IN HELIX WAVEGUIDE

Mode	$\mathrm{Real}\,\frac{Bk_n^2}{\Gamma_1 - \Gamma_n}$
TE_{11}	0.713
TM_{11}	8.79
TE_{12}	−8.05
$\Sigma =$	1.55

4.5 *Curved Beam Guide*

Let us consider a curved beam guide made of a sequence of confocal lenses propagating the fundamental gaussian mode. The radius of curvature R_c, the wavelength λ, and the beam size w are found, with the help of equation (50), to be related to the maximum power conversion P_c by

$$R_c = \frac{\pi^2 w_o^3}{\lambda^2 (P_c)^{\frac{1}{2}}}. \tag{77}$$

As in a previous example, the width of the guide containing 95 per cent of the power in the wanted mode is $a = 2w_o$; therefore,

$$R_c = \frac{1.23}{(P_c)^{\frac{1}{2}}} \frac{a^3}{\lambda^2}. \tag{78}$$

This exact result compares with the approximate value from equation (33) with $w = a$ and $p = 1$,

$$R = \frac{1.36}{(P_c)^{\frac{1}{2}}} \frac{a^3}{\lambda^2}. \tag{79}$$

Considering that the exact value relates to an open lens waveguide and the approximate one relates to a hollow metallic rectangular waveguide, the agreement is excellent.

V. DISCUSSION AND CONCLUSION

The direction-changing capability of electromagnetic waveguides may be limited by (*i*) radiation, if the guided field extends into an open freely propagating region, and (*ii*) mode conversion. Radiation is the limitation for single-mode open guides that have transverse fields extending indefinitely into a freely propagating region. An estimate of permitted bending radius may be made by using equations (15) or (3) and the knowledge of the field for the straight guide. For a straight guide transverse field decaying exponentially $[\exp(-x/\xi)]$, the radiation attenuation coefficient in a bend of radius R was found to be of the form

$$\alpha_R = c_1 \exp(-c_2 R), \tag{13}$$

where c_1 and c_2 are large constants. As a result, α_R is large for

$$R < \frac{1}{c_2} \log c_1 \tag{14}$$

and small for R greater than that value.

When the guide supports higher order modes, mode conversion loss tends to be the controlling factor. In Section III formulas are developed for permissible bend radius R versus transverse field width a, the guided wavelength λ_g, and fractional power P_c lost to other modes. Numerous possible cases are treated, depending on the relation between the mode coupling coefficients k, the signal mode propagation coefficient $\Gamma_1 = \alpha_1 + i\beta_1$, and the propagation coefficient of the mode coupled to, in the bend $\Gamma_2 = \alpha_2 + i\beta_2$. A case which should be very common is one of small or moderate losses and gradual bends:

$$|k^2| \ll (\beta_1 - \beta_2)^2, \tag{28}$$

$$(\alpha_1 - \alpha_2)^2 \ll (\beta_1 - \beta_2)^2, \tag{29}$$

$$\alpha_2 L \ll 1, \tag{30}$$

where L is the length of the bend. Then an approximation for the bend radius permitted is

$$R = \frac{4.1}{(2p \pm 1)(P_c)^{\frac{1}{2}}} \frac{a^3}{\lambda^2}, \tag{33}$$

and for the permitted abrupt tilt angle δ

$$\delta = 0.62\,(P_t)^{\frac{1}{2}} \frac{\lambda_g}{a}, \tag{24}$$

in which p is the number of extrema in the transverse field distribution. Examples are given in Sections 4.1 through 4.4 which show that known theory for several hollow metallic and open lens waveguides agree well with these expressions.

One must use caution in applying these expressions to new waveguides where the modes coupled in the bend are not known and, more importantly, where the phase constant differences are not known. If by design or misfortune a degeneracy exists between modes coupled by the bend, $\beta_1 = \beta_2$, a radically more severe restriction on bend R occurs. Sections 3.6 and 3.7 discuss this situation. However, since β's are large compared with typical α's, it usually is possible to avoid these restrictive conditions and justify equations (28) and (29) by small modifications of the guiding structure.

If the mode coupled to is very lossy, so that $\alpha_2 L \gg 1$, equation (33) does not hold. Section 3.5 and equation (38) relate to this case. We cite one example in Section 5.4 which supports equation (38); but more experience with coupling to lossy modes is needed.

APPENDIX A

Supplement to Section II

We note that the maintenance of equiphase differences on $F(x)$ for all x, but on radial planes differing by $\Delta\varphi$ (Fig. 1), requires

$$k_s R\ \Delta\varphi \geqq k_z(R + x_r)\ \Delta\varphi, \tag{80}$$

where k_s is the phase constant for a plane wave in the region surrounding the waveguide. For the equal sign in equation (80) a plane wave in the x_r region is traveling at the velocity of light and equation (80) yields

$$x_r = \frac{(k_z - k_s)}{k_s} R. \tag{81}$$

The energy traveling at $x > x_r$ is presumed lost to radiation, since to remain guided would imply energy traveling at greater than the velocity of light. The fraction of the total energy in the cross section at $x > x_r$ is $\mathcal{E}_l/\mathcal{E}_T$, where $\mathcal{E}_l$ and $\mathcal{E}_T$ are given by equations (4) and (5). How rapidly, as a function of distance along the direction of propagation, does energy flow out from the main energy packet to this region at $x > x_r$? For a wave in an infinite uniform medium the energy remains collimated for a distance

$$z_c = \frac{a^2}{2\lambda_s}, \tag{82}$$

where a is the transverse field width and λ_s is the wavelength in that medium. It may be expected that an approximate distance z_c would be required for energy to flow out from the guided field of the same width a. Noting a power decay rate $e^{-2\alpha z} \approx 1 - 2\alpha z$, the fractional power loss becomes

$$\frac{\mathcal{E}_l}{\mathcal{E}_T} = 2\alpha_r z_c \tag{83}$$

or

$$\alpha_r = \frac{1}{2z_c}\frac{\mathcal{E}_l}{\mathcal{E}_T}. \tag{84}$$

Numerical Evaluations of a Specific Case

The potential usefulness of equation (3) is in estimating radiation losses of curved open waveguides for which the straight-guide fields are known, but for which a solution in the curved coordinate system is not known. Here we compare the results of using equation (3) with the results of a direct solution, to obtain an indication of the accuracy that might be expected in other cases. The case is defined by equations (8) and (9), which lead to equations (10), (11), and (12) for $\mathcal{E}_l$, $\mathcal{E}_T$, and z_c.

We provide additional expressions needed in the numerical calculations: from known theory[5,16]

$$\frac{1}{\xi} = [k^2(n_1^2 - n_3^2) - k_x^2]^{\frac{1}{2}} \tag{85}$$

where k is the free space wave number, n_1 is the index of refraction of the dielectric slab, and n_3 is in the index of the surrounding region. The quantity k_x may be obtained graphically as a function of t/A and is reproduced here in Fig. 5, from Ref. 5. The quantity A is the value of t at which the second propagating mode appears,

$$A = \frac{\pi}{k(n_1^2 - n_3^2)^{\frac{1}{2}}} \frac{\lambda}{2(n_1^2 - n_3^2)^{\frac{1}{2}}}. \tag{86}$$

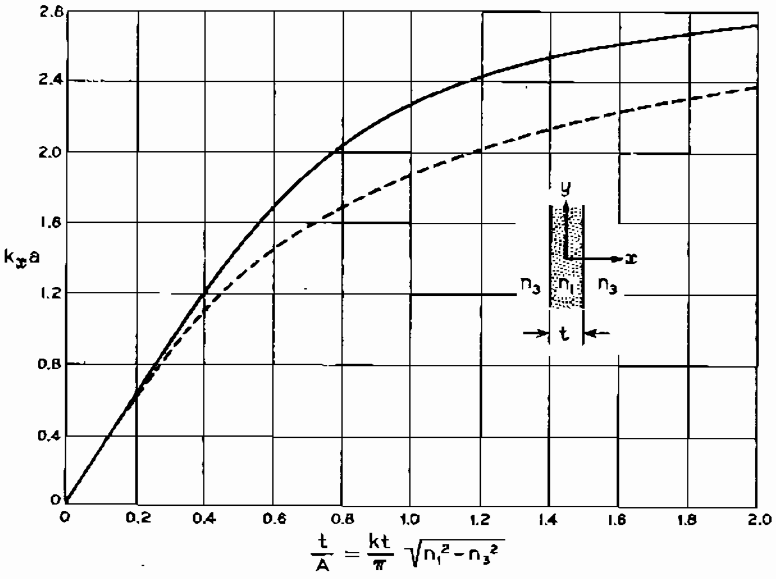

Fig. 5 — Normalized transverse wave number $k_x a$ versus normalized thickness t/A for a two-dimensional dielectric waveguide. ——— fundamental mode polarized perpendicular to the dielectric sheet and $n_3 = n_1/1.5$; – – – – fundamental mode polarized perpendicular to the sheet and $n_1/n_3 - 1 \ll 1$, or fundamental mode polarized parallel to the sheet and n_3 arbitrary.

Also known are [5,9]

$$k_z^2 = n_1^2k^2 - k_x^2 , \tag{87}$$

$$k_z^2 = n_3^2k^2 + \frac{1}{\xi^2} , \tag{88}$$

and in the region of considerable interest where

$$k_x \ll kn_1 , \tag{89}$$

the approximations

$$k_z \doteq kn_1 - \frac{1}{2}\frac{k_x^2}{kn_1} \tag{90}$$

$$k_z - kn_3 \equiv k_z - k_c = k(n_1 - n_3) - \frac{1}{2}\frac{k_x^2}{kn_1} \tag{91}$$

are valid. Using the above relations, one can calculate α_r, given t, λ, n_1, and n_3.

Table IV lists the principal parameters and a comparison with more exact theory for several cases. In Table IV the first five columns define the waveguide; c_1 and c_2 are values from equation (13), found in turn by evaluating equations (3) through (7). The table also lists the radiation attenuation coefficient α_r, the estimate of the associated bend radius R, the value of R from Ref. 5, and the ratio. The estimate from equation (3) is consistently lower than the true required R (in the approximate ratio 0.6) for a wide range of index differences $(n_1 - n_3)$ and bend radii R.

The table also lists the transverse distance x_r at which the velocity of light condition occurs. It is interesting that it is so close to the waveguide.

Additional support for the approximate calculation based on equation (3) comes from an additional case. It is readily verified from exact theory that the case 1 condition, $n_1 = 1.5$ and $n_3 = 1.0$, yields different radiation losses for the two polarizations of wave if the thickness t is fixed. However, if t is adjusted to give the same external field decay constant ξ of equation (9), then the radiation losses are the same for the two polarizations of wave.

APPENDIX B

Solutions of the Coupled-Wave Equations (26) and (27)

If one assumes that the coupling coefficient k in equation (26) and (27) is pure imaginary, $k = ic$, one can express the fractional power

TABLE IV—TABULATION OF IMPORTANT PARAMETERS IN CURVED DIELECTRIC WAVEGUIDES

Case	t (10^{-6} m)	t/λ	n_1	n_3	c_1	c_2	R (meter)	α_r (neper/m)	x_r (10^{-6} m)	R from Ref. 5 for same α_r (meter)	Ratio = col (8)/col (11)
1*	0.198	0.7	1.5	1.0	2.57×10^4	3.47×10^9	3.54×10^{-6}	11.6	0.846	5.49×10^{-6}	0.645
							6.15×10^{-3}	1.34×10^{-4}	1.47	11×10^{0}	0.656
2†	0.372	0.25	1.5	1.485	0.46×10^6	2.570	4.17×10^{-3}	1.0	6.5	6.89×10^{-3}	0.606
3†	1.04	0.7	1.5	1.485	1.037×10^6	1.46×10^4	0.807×10^{-3}	0.776	4.0	1.18×10^{-3}	0.68
							1.43×10^{-3}	0.895×10^{-4}	7.09	2.37×10^{-3}	0.60
4†	1.79	1.20	1.5	1.485	1.46×10^6	2.55×10^4	0.355×10^{-3}	16.9	2.55	0.593 ± 10^{-3}	0.6
							0.442×10^{-3}	1.89	3.18	0.711×10^{-3}	0.62
							0.6×10^{-3}	0.0321	4.31	0.948×10^{-3}	0.63
							0.938×10^{-3}	5.97×10^{-4}	6.75	1.18×10^{-3}	0.79
5†	2.38	1.6	1.5	1.485	2.18×10^6	3.04×10^4	0.336×10^{-3}	7.9	2.7	0.593×10^{-3}	0.566
							0.423×10^{-3}	0.368	3.4	0.711×10^{-3}	0.59
							0.585×10^{-3}	3.97×10^{-3}	4.7	0.948×10^{-3}	0.617
6†	1.18	0.25	1.5	1.4985	0.543×10^4	81.4	0.106	1.0	16.5	0.18	0.59

* The electric field is parallel to the dielectric slab.
† Applies for either polarization.

P_c converted out of the signal (that is case 1) mode as

$$P_c = 1 - \frac{\exp[(\alpha_1 - \alpha_2)z]}{4} \left| \left[1 - \frac{1}{(1+\kappa^2)^{\frac{1}{2}}}\right] \exp\left[i\left(1 + \frac{1}{\kappa^2}\right)^{\frac{1}{2}} cz\right] + \left[1 + \frac{1}{(1+\kappa^2)^{\frac{1}{2}}}\right] \exp\left[-i\left(1 + \frac{1}{\kappa^2}\right)^{\frac{1}{2}} cz\right] \right|^2, \quad (92)$$

where

$$\kappa = \frac{i2c}{\Gamma_1 - \Gamma_2} \quad (93)$$

and

$$\Gamma_{\substack{1\\2}} = \alpha_{\substack{1\\2}} + i\beta_{\substack{1\\2}} . \quad (94)$$

In these formulas, Γ_1 and Γ_2 are the propagation constants of the wanted and spurious modes, respectively; in general, they are complex and their real parts, α_1 and α_2, are the attenuation constants; their imaginary parts, β_1 and β_2, are the phase constants. We bear in mind that $k = ic$ has only been proven valid in lossless waveguides, and for one case of coupling to a lossy mode (helix waveguide) k is complex.

Another useful expression is for the signal wave amplitude E_1 when the coupling k is small compared with $(\Gamma_1 - \Gamma_2)$, or more specifically,

$$|4k^2| \ll (\Gamma_1 - \Gamma_2)^2 \quad (95)$$

and

$$|k^2| \ll |\Gamma_2(\Gamma_1 - \Gamma_2)| . \quad (96)$$

Then we may write

$$E_1 \cong \exp(-\Gamma_1 z)\left\{\left[1 - \frac{k^2}{(\Gamma_1 - \Gamma_2)^2}\right] \exp\left[-\frac{k^2 z}{(\Gamma_1 - \Gamma_2)}\right] + \frac{k^2}{(\Gamma_1 - \Gamma_2)^2} \exp[(\Gamma_1 - \Gamma_2)z]\right\}. \quad (97)$$

The first term corresponds to the low-loss normal mode of the coupled region, and the second term to the high-loss mode (we assume $\alpha_2 > \alpha_1$).

For Section 3.4, it is valid to take $k = ic$; equation (92) yields a conversion loss of

$$P_c = \left(\frac{2c}{\beta_1 - \beta_2}\right)^2 \sin^2\left[\frac{(\beta_1 - \beta_2)z}{2}\right]. \quad (98)$$

For Section 3.5 we use equation (97), keeping a complex k; note that for $\alpha_2 L \gg 1$, only the first term remains significant and the propagation constant of the normal mode is

$$\Gamma_1 + \frac{k^2}{(\Gamma_1 - \Gamma_2)}. \quad (99)$$

This yields equation (36) for α_B, the added attenuation resulting from the bend.

For Section 3.7 we again use equation (97); the first term predominates with the assumption

$$\frac{k^2}{(\Gamma_1 - \Gamma_2)} \ll 1 \quad (100)$$

and equation (99) yields equation (48).

For Section 3.6, the case of low-loss modes degenerately coupled, equation (92) yields

$$P_c \cong 1 - \exp[(\alpha_1 - \alpha_2)z] \left| \cos\left(cz + \frac{i}{\kappa}\right) \right|^2. \quad (101)$$

It is also well known that the signal amplitude is given by[2-4]

$$|E_1| = |\cos cz|, \quad (102)$$

the undesired mode amplitude by

$$|E_2| = |\sin cz| \quad (103)$$

and the fractional conversion loss P_c by

$$P_c = \sin^2 cz. \quad (104)$$

APPENDIX C

Supplementary Information Concerning the Derivation of Equation (22)

Carrying out the integration of equation (16) for the rectangular metallic waveguide as outlined in Section 3.1 yields a conversion loss resulting from the tilt of

$$P_t = B\left(\frac{\delta w}{\lambda_c}\right)^2, \quad (105)$$

where

$$B = \frac{\pi^2}{3}\left[1 \pm \frac{\left(\frac{p}{w}\right)^2 - \left(\frac{q}{b}\right)^2}{\left(\frac{p}{w}\right)^2 + \left(\frac{q}{b}\right)^2}\frac{6}{\pi^2 p^2}\right]. \tag{106}$$

The + or − sign corresponds to the TE_{pq} or TM_{pq} modes, respectively.

For the lowest order TE mode, $p = 1$ and $q = 0$, B becomes 5.28. For the TE or TM mode with $p = 1$ and $q = 1$, B ranges from 5.28 to 1.28 as the dimensions of the guide vary between $w \ll b$ and $w \gg b$. The limits on B are 5.28 and 1.28 for any p or q. We somewhat arbitrarily chose a value $(5.28 \times 1.28)^{1/2} = 2.6$ to represent all modes simultaneously.

REFERENCES

1. Miller, S. E., "Directional Control in Light-Wave Guidance," B.S.T.J., *43* No. 4 (July 1964), pp. 1727–1739.
2. Jouguet, M., "Effects of the Curvature on the Propagation of Electromagnetic Waves in Guides of Circular Cross Section," Cables and Transmission (Paris), *1*, No. 2 (July 1947), pp. 133–153.
3. Miller, S. E., "Notes on Methods of Transmitting the Circular Electric Wave Around Bends," Proc. I.R.E., *40*, No. 5 (September 1952), pp. 1104–1113.
4. Miller, S. E., "Coupled Wave Theory and Waveguide Applications," B.S.T.J., *33*, No. 3 (May 1954), pp. 661–719.
5. Marcatili, E. A. J., "Bends in Optical Dielectric Guides," B.S.T.J., this issue, pp. 2103–2132.
6. Miller, S. E., "Integrated Optics: An Introduction," B.S.T.J., this issue, pp. 2059–2069.
7. Schelkunoff, S. A., "Conversion of Maxwell's Equations into Generalized Telegraphist's Equations," B.S.T.J., *34*, No. 5 (September 1955), pp. 995–1043.
8. Unger, H. G., "Normal Mode Bends for Circular Electric Waves," B.S.T.J., *36*, No. 5 (September 1957), pp. 1292–1307.
9. Collin, R. E., *Field Theory of Guided Waves*, New York: McGraw Hill, 1960.
10. Rowe, H. E., and Warters, W. D., "Transmission in Multi-Mode Waveguide with Random Imperfections," B.S.T.J., *41*, No. 3 (May 1962), pp. 1031–1170.
11. Miller, S. E., "On Solutions for Two Waves with Periodic Coupling," B.S.T.J., *47*, No. 8 (October 1968), pp. 1801–1822.
12. Marcatili, E. A. J., "Effects of Redirectors, Refocusers, and Mode Filters on Light Transmission through Aberrated and Misaligned Lenses," B.S.T.J., *46*, No. 8 (October 1967), pp. 1733–1752.
13. Unger, H. G., "Lined Waveguide," B.S.T.J., *41*, No. 2 (March 1962), pp. 745–768.
14. Unger, H. G., "Normal Modes and Mode Conversion in Helix Waveguide," B.S.T.J., *40*, No. 1 (January 1961), p. 255; also "Helix Waveguide Theory and Application," B.S.T.J., *37*, No. 6 (November 1958) pp. 1599–1647.
15. Morgan, S. P., and Young, J. A., "Helix Waveguide," B.S.T.J, *35*, No. 6 (November 1956), pp. 1347–1384.
16. Young, D. T., "Measured TE_{01} Attenuation in Helix Waveguide with Controlled Straightness Deviations," B.S.T.J., *44*, No. 2 (February 1965), pp. 273–282.