APPENDIX IV

Time Constant for the Capacity of the Transition Region

For this case we shall consider the case of holes in an a-c. field with potential

$$\psi = \frac{kT}{q}\left(\frac{x}{L_r} + \frac{xe^{i\omega t}}{L_1}\right)$$

where the d-c. retarding field is kT/qL_r and the a-c. field is kT/qL_1 where $1/L_1$ is considered small for the linear theory presented here. The expression for the current of holes is

$$-D\frac{\partial p}{\partial x} - \mu p\frac{\partial \psi}{\partial x} = -D\left[\frac{\partial p}{\partial x} + p\left(\frac{1}{L_r} + \frac{e^{i\omega t}}{L_1}\right)\right]$$

We shall obtain a solution for p by letting

$$p = p_0 e^{-x/L_r} + p_1[e^{-x/L_r} - e^{-\gamma x}]e^{i\omega t},$$

while neglecting recombination in this region so that p must satisfy the condition $\dot{p} = -\partial$ (hole current)$/\partial x$ leading to the differential equation

$$D\left[\frac{\partial^2 p}{\partial x^2} + \frac{\partial p}{\partial x}\left(\frac{1}{L_r} + \frac{e^{i\omega t}}{L_1}\right)\right] - \dot{p} = 0$$

There are three separate exponential dependencies of the variables leading to three equations (neglecting terms of order $(1/L_1)^2$)

e^{-x/L_r}: $$D\left[p_0\frac{1}{L_r^2} - p_0\frac{1}{L_r^2}\right] = 0$$

$e^{-x/L_r+i\omega t}$: $$D\left[p_1\frac{1}{L_r^2} - p_1\frac{1}{L_r^2} - \frac{1}{L_r L_1}p_0\right] - i\omega p_1 = 0$$

$e^{-\gamma x+i\omega t}$: $$D[\gamma^2 - \gamma/L_r]p_1 - i\omega p_1 = 0$$

The first equation is satisfied by the equilibrium distribution and the second by

$$p_1 = -p_0 D/i\omega L_1 L_r$$

and the last by

$$\gamma = \frac{1 + \sqrt{1 + 4i\omega L_r^2/D}}{2L_r}$$

It is evident that dispersive effects set in when

$$\omega = D/4L_r^2$$

This corresponds to the result used in (4.31) in which $(x_{Tn} - x_{Tp})/10$ was used for L_r. For smaller values of ω the current may be calculated and put in simple form by expanding γ up to terms including ω^2. The resulting expression for the current is

$$I = -i\omega q\, p_0\, L_r(L_r/L_1)e^{i\omega t}$$

This is interpreted as follows: The a-c. voltage across a layer L_r thick is

$$\delta\psi = (kT/q)\ (L_r/L_1)e^{i\omega t}$$

and, if we consider plus voltage as producing a field from left to right, then the a-c. voltage across L_r is $V = -\delta\psi$. Substituting this for $(L_r/L_1)\exp(i\omega t)$ gives

$$I = i\omega q p_0\, L_r(q/kT)V$$

Here qp_0L_r is the total charge in the layer L_r, (qV/kT) is an average fractional change in this charge for V so that $(qp_0L_r)\ (qV/kT) \div V$ is a capacity.

APPENDIX V

The Effect of Surface Recombination

In this appendix we shall consider the effect of surface recombination upon the characteristics of the p-n junction. As for Section 4 we shall illustrate the theory for the case of holes diffusing into n-type material. For simplicity we shall treat a square cross-section bounded by $y = \pm w$, $z = \pm w$, the current flow being along $+x$.

We shall denote the a-c. component of p as

$$p_1 \equiv p_1\,(x, y, z, t)$$

At $x = 0$, the edge of the n-region, we shall suppose that φ_p and ψ are independent of y and z so that we shall have

$$p_1(0, y, z, t) = p_{10}\, e^{i\omega t} = (p_n\, qv_1/kT)e^{i\omega t}$$

by reasoning similar to that used for equation (4.5). The boundary condition at the surface will be

$$-D\frac{\partial p_1}{\partial y} = sp_1 \qquad \text{for } y = +w$$

This states that the recombination per unit area is sp_1 and is equal to the diffusion to the surface $-D\partial p_1/\partial y$. Similar boundary conditions hold for the other surfaces. By standard procedures involving separation of variables we may verify that the solution satisfying the boundary conditions is

$$p_1 = \sum_{i,j=0}^{\infty} a_{ij}\, e^{-\alpha_{ij}x + i\omega t} \cos\beta_i y \cos\beta_j z$$

where the eigenvalues β_i are determined by the boundary condition

$$\beta_i w \tan \beta_i w = sb/D \equiv \chi.$$

We use $\theta_i = \beta_i w$ for brevity later. Because of the symmetry of the boundary conditions it is not necessary to include sine functions in the sum. The value of α_{ij} is given by

$$\alpha_{ij} = (1 + i\omega\tau_{ij})^{\frac{1}{2}}/(D\tau_{ij})^{\frac{1}{2}}$$

where τ_{ij} is the lifetime of a hole in the eigenfunction $\cos \beta_j y \cos \beta_j z$; i.e. τ_{ij} is the lifetime which makes

$$p = \exp(-t/\tau_{ij}) \cos \beta_i y \cos \beta_j z,$$

a function which satisfies the surface boundary conditions, a solution of the equation

$$\partial p/\partial t = D\nabla^2 p - p/\tau = -D(\beta_i^2 + \beta_j^2)p - p/\tau$$

where to simplify the subsequent expressions we have omitted the subscript p from τ. This equation leads to

$$\frac{1}{\tau_{ij}} = D(\beta_i^2 + \beta_j^2) + \frac{1}{\tau}.$$

The coefficients a_{ij} are readily found since the $\cos \beta_i y$ functions form an orthogonal set (as may be verified by integrating by parts and using the boundary conditions). The values are

$$a_{ij}/p_{10} = 4[\sin \theta_i \sin \theta_j]/\theta_i\theta_j[1 + (1/2\theta_i) \sin 2\theta_i]\cdot[1 + (1/2\theta_j) \sin 2\theta_j]$$

The current corresponding to this solution is

$$I_1 = -qD \iint (\partial p/\partial x)\, dy\, dz$$

integrated over the cross section at $x = 0$. This gives

$$I_1 = qDp_{10}e^{i\omega t}\, \Sigma\, \alpha_{ij}(a_{ij}/p_{10})(4w^2/\theta_i\theta_j) \sin \theta_i \sin \theta_j$$

Substituting for a_{ij} and inserting $p_{10} = p_n q v_1/kT$, we obtain an expression for the admittance $A_p = I_1/V_1 \exp(i\omega t)$:

$$A_p = 4w^2 q\mu p_n \Sigma_{ij} \alpha_{ij} \frac{4 \sin^2 \theta_i \sin^2 \theta_j}{\theta_i^2 \theta_j^2 \left[1 + \left(\frac{1}{2\theta_i}\right) \sin 2\theta_i\right]\left[1 + \left(\frac{1}{2\theta_j}\right) \sin 2\theta_j\right]}$$

where the sum plays the role formerly taken by $(1 + i\omega t)^{1/2}/\sqrt{D\tau}$ in equation (4.12); the factor $4w^2$ is the area of the junction.

We shall analyze the formula for the case in which recombination on the surface is smaller than diffusion to the surface so that χ is not large. The values of θ_i, over which the sum is to be taken, may be estimated as follows: in each interval of θ_i of the form $n\pi$ to $(n + (\frac{1}{2}))\pi$, $\theta_i \tan \theta_i$ varies from 0 to ∞, giving one solution to $\theta_i \tan \theta_i = \chi$. For χ small, the solutions are approximately

$$\theta_0 \doteq \sin \theta_0 \doteq \tan \theta_0 \doteq \sqrt{\chi}$$

$$\theta_1 \doteq \pi + \chi/\pi; \; - \sin \theta_1 \doteq \tan \theta_1 \doteq \chi/\pi$$

$$\cdots \qquad \cdots$$

$$\theta_n \doteq n\pi + \chi/n\pi; \; (-1)^n \sin \theta_n \doteq \tan \theta_n \doteq \chi/n\pi$$

From this we see that the terms in the sum are as follows:

$$\alpha_{00} \cdot 4\chi^2/\chi^2 4 = \alpha_{00}$$

$$\alpha_{n0} \cdot 2(\chi/n\pi)^2/(n\pi)^2 = \alpha_{n0} 2\chi^2/n^4\pi^4$$

$$\alpha_{nm} \cdot 4\chi^4/n^4 m^4 \pi^8$$

From this it is evident that unless χ is large, the series converges very rapidly. (This conclusion is not altered when the increase in α_{nm} with $\beta_n\beta_m$ is considered.) Thus the dominant term in the admittance is

$$4w^2 q\mu p_0 \, (1 + i\omega\tau_{00})^{1/2}/\sqrt{D\tau_{00}}$$

where

$$1/\tau_{00} = 2\left(\frac{D}{w^2}\right)(\theta_0^2) + 1/\tau$$

$$\doteq 2\left(\frac{D}{w^2}\right)\frac{sw}{D} + 1/\tau$$

$$= 2\left(\frac{s}{w}\right) + 1/\tau$$

This expression is valid only for sw/D small so that $\theta_0^2 \doteq sw/D$. The term $s/(w/2)$ represents the rate of decay due to holes recombining on the surface, s having the dimensions of velocity. For $\omega \gg 1/\tau_{00}$, the admittance becomes $4w^2 q\mu p_0(i\omega/D)^{1/2}$, the same value as given in equation (4.12) for large ω and an area $4w^2$.

The conclusion from this appendix is that for χ small, the effect of surface recombination is simply to modify the effective value of τ and otherwise leave the theory of Section 4 unaltered.

For very large values of χ, it is necessary to consider higher terms in the sum and several values of τ will be important. Under these conditions the approximation is that, at $x = 0$, p_1 is independent of x and y may become a poor one, especially for forward currents, because the transverse currents to the edges will be important. Under these conditions the role of surface recombination will give rise to patch effects of the sort discussed in Section 4.

APPENDIX VI

The Effect of Trapping Upon the Diffusion Process

In this appendix we shall investigate the effect of the trapping of holes upon the impedance. We denote the density of mobile holes in the valence-bond band by p and the density of holes trapped in acceptors by p_a. For thermal equilibrium at room temperature there will be an equilibrium ratio, called α, for p_a/p. For germanium $\alpha \doteq 10^{-4}$ and for silicon $\alpha \doteq 0.1$ to 0.2.

We shall consider four processes which occur at rates (per particle per unit time) as follows:

ν_r direct recombination of a hole with an electron (free or bound to a donor)
ν_t trapping of a hole by an acceptor
ν_{ra} recombination of a hole trapped on an acceptor
ν_e excitation of a trapped hole into the valence-bond band.

Under equilibrium conditions as many holes are being trapped (rate $p\nu_t$) as are being excited ($p_a\nu_e$): hence $\nu_t = \alpha\nu_e$.

We shall study solutions of the customary form for the a-c. components:

$$p_1 = p_{10}\, e^{i\omega t - \gamma x}$$

$$p_{1a} = p_{1a0}\, e^{i\omega t - \gamma x}$$

These must satisfy the equations

$$\dot{p}_1 = D\nabla^2 p_1 - (\nu_t + \nu_r)p_1 + \nu_e p_{1a}$$

$$\dot{p}_{1a} = \nu_t p_1 - (\nu_e + \nu_{ra})p_{1a}$$

These lead readily to the equation for γ:

$$D\gamma^2 = i\omega + \nu_r + \nu_t - \nu_e\nu_t/(i\omega + \nu_e + \nu_{ra}) = i\omega \cdot \left[1 + \frac{\nu_e\nu_t}{(\nu_e + \nu_{ra})^2 + \omega^2}\right] + \nu_r + \nu_t\left[1 - \frac{\nu_e}{(\nu_e + \nu_{ra}) + \omega^2/(\nu_e + \nu_{ra})}\right]$$

From this equation we can directly reach the important conclusion that the trapping process can never lead to a capacitative term larger than the resistive term. This result is obtained by analyzing the complex phase of γ, the admittance being proportional to γ. In particular, we find that the real term in $D\gamma^2$ is always positive, as may be seen from inspection, so that the complex phase angle of γ is less than 45°.

The form reduces to a simple expression if ν_e and ν_t are very large compared to ν_r, ν_{ra} and ω, a situation which insures local equilibrium between p and p_α. Under these conditions we obtain

$$D\gamma^2 = i\omega[1 + \alpha] + \nu_r + \alpha\nu_{ra}$$

Dividing by $(1 + \alpha)$ gives

$$[D/(1 + \alpha)]\gamma^2 = [Dp/(p + p_a)]\gamma^2 = i\omega + \frac{p\nu_r + p_a\nu_{ra}}{p + p_a}$$

The interpretation is that the holes diffuse as if their diffusion constant were reduced by the fraction of the time $p/(p + p_a)$ they are free to move and recombine with a properly weighted average of ν_r and ν_{ra}.

APPENDIX VII

Solutions of the Space Charge Equation

We shall first show that the space charge equation (2.11) has a unique solution for the one dimensional case. For simplicity we write (2.11) in the form

$$\frac{d^2u}{dx^2} = \sinh u - f(x) \tag{A7.1}$$

to which it can be readily reduced. We shall deal with the case for which

$$f = f_a \text{ for } x < x_a \tag{A7.2}$$

$$f = f_b \text{ for } x > x_b > x_a \tag{A7.3}$$

so that the interval (x_a, x_b) is bounded by semi-infinite blocks of uniform semiconductor. We shall require that u be finite at $x = \pm\infty$. This boundary condition requires that for large values of $|x|$

$$u = u_a + A_a e^{+\gamma_a x} \qquad x \to -\infty \tag{A7.4}$$

$$u = u_b + A_b e^{-\gamma_b x} \qquad x \to +\infty \tag{A7.5}$$

where

$$\sinh u_a = f_a, \qquad \sinh u_b = f_b$$

$$\gamma_a = |(\cosh u_a)^{1/2}|, \qquad \gamma_b = |(\cosh u_b)^{1/2}|$$

(If the opposite signs of the γ's were present, the boundary conditions would not be satisfied.) The exponential solutions are valid for $|u - u_a|$ or $|u - u_b| \ll 1$. For larger values, however, solutions exist which are obtained by integrating (A7.1) to larger or smaller values of x.

For these extended solutions the values of $u(x, A_a)$ and $u'(x, A_\alpha)$ $(= du/dx)$

are monotonically increasing functions of A_a. This may be seen by considering $x = x_a$. For A_a sufficiently small, the value of $u(x_a, A_a)$ and $u'(x_a, A_a)$ are given simply by (A7.4). For larger values of A_a, an exact integral will be required. It is evident, however, that l solutions of the form (A7.4) are related simply by translation for $x < x_a$. Hence increasing A_a is simply equivalent to integrating (A7.1) to larger values of x and it is evident that this increases u and u' mo otonically. It may be verified that for a sufficiently large A_a the solution becc es infinite at x_a so that $u(x_a, A_a)$ $u'(x_a, A_a)$ both vary monotonically and c tinuously from $-\infty$ to $+\infty$ as A_a varies from negative to positive valu We shall refer to this property of $u(x_a, A_a)$, $u'(x_a, A_a)$ as P_1.

We next wish to show that $u(x_1, A_a)$, $u'(x_1, A_a)$ has the property P_1 for values of $x_1 > x_a$. To prove this we note that if for any x_1, $u(x_1, A_a)$ and $u'(x_1, A_a)$ are finite, the solution may be integrated somewhat further to obtain $u(x_2, A_a)$, $u'(x_2, A_a)$ for $x_2 > x_1$. From equation (A7.1) it is evident that an increase in either $u(x_1, a)$ or $u'(x_1, a)$ will result in an increase in d^2u/dx^2 in the interval $x_1 < x < x_2$ so that u and u' at x_2 are monotonically increasing functions of u and u' at x_1. Hence if u and u' at x_1 have the property P_1, so do u and u' at x_2. By extending this argument we conclude that u and u' at any value of x have the property P_1. (A rigorous proof can easily be completed along these lines provided that $|f(x)|$ is finite.)

Similarly it may be shown, starting from (A7.5), that $u(x, A_b)$ is a monotonically increasing function of A_b and $u'(x, A_b)$ is a monotonically *decreasing* function of A_b.

In order to have a solution satisfying (A7.4) and (A7.5) we must have, for any selected point x,

$$u(x, A_a) = u(x, A_b) \tag{A7.6}$$

$$u'(x, A_a) = u'(x, A_b) \tag{A7.7}$$

Now as the equation $u(x, A_a) = u(x, A_b)$ varies from $-\infty$ to $+\infty$, $u'(x, A_a)$ varies from $-\infty$ to $+\infty$ and $u'(x, A_b)$ varies from $+\infty$ to $-\infty$, monotonically and continuously. Hence there is one and only one solution of (A7.1) satisfying (A7.4) and (A7.5).

In order to verify that the solutions discussed in Section 2 are correct for large and for small K, we show schematically in Fig. A1 the solution for a representative K as a dashed line together with the curve $u = u_0(y) = \sinh^{-1} y$. In terms of u_0, equation (2.16) becomes

$$\frac{d^2u}{dy^2} = \frac{1}{K^2}(\sinh u - \sinh u_0). \tag{A7.8}$$

From the symmetry of the equation, it is evident that u must be an odd function of y and hence that the solution must pass through the origin. The boundary condition in this case will be that $u \rightarrow u_o$ for $y \rightarrow \pm\infty$ so that there will be no space charge far from the junction. We can conveniently use the origin as the point at which the solution from $y = +\infty$ joins that from $y = -\infty$; from symmetry, this requires merely that $u = 0$ when $y = 0$.

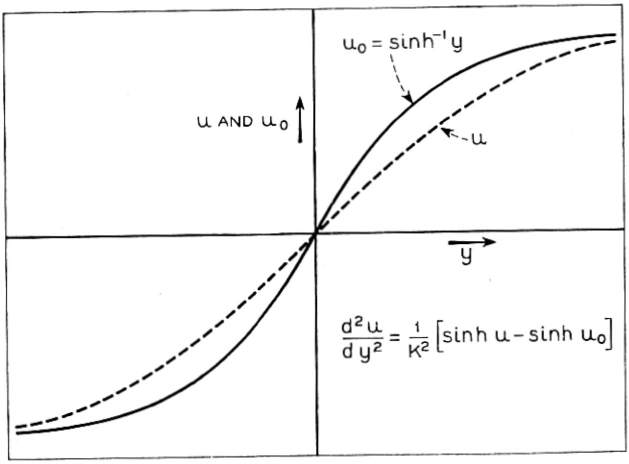

Fig. A1—Behavior of the solution of Equation (2.16) or (A7.8).

For large negative y, $u = \sinh^{-1} y$ and $du/dy = 1/\cosh u_0$ so that du/dy is small. It is at once evident that, for large values of K, u must lie above u_0 so that the integral

$$(1/K^2) \int_{-\infty}^{y} (\sinh u - \sinh u_0)\, dy = \frac{du}{dy} \qquad \text{(A7.9)}$$

will be large enough to make the solution $u(y)$ pass through the origin. If $u - u_0 > 2$ over the region of largest difference, the space charge will be largely uncompensated and the solution will correspond to that used in equation (2.18). On the other hand, as $K \rightarrow 0$, the requirement that $u(y)$ pass through the origin leads to the conclusion that $u - u_0$ must be small for all values of y. The possibility that u oscillates about u_0 need not be considered since it may readily be seen that, if for any negative value of y, say y_1, both $u(y_1)$ and $u'(y_1)$ are less than $u_0(y_1)$ and $u'(y_1)$, then $u(y)$ and $u'(y)$ are progressively less than $u_0(y)$ and $u_0'(y)$ as y increases from y_1 to 0. Hence, if for negative y the u curve goes below the u_0 curve, it cannot pass through the origin.

APPENDIX VIII

List of Symbols

(Numbers in parentheses refer to equations)

$a = (N_d - N_a)/x$ (2.14)
A = admittance per unit area of junction (4.23)
A_p = component of A due to hole flow into n-region (4.12) (4.24)
A_n = component of A due to electron flow into p-region (4.25)
A_T = component of A due to varying charge distribution in transition region
A also used as a constant coefficient in various appendices
b = ratio of electron mobility to hole mobility
b = symbol for base in Sections 5 and 6
B constant coefficient in various expansions in appendices
c = symbol for collector in Section 6; a length in Appendix III
C = capacity per unit area
C_n, C_p (4.25) (4.27) as for A_n, A_p
C_T (2.42) (2.45) (2.56) as for A_T
D = diffusion constant for holes (bD is the diffusion constant for electrons)
$e = 2.718\ldots$
f see Appendix 7
g = rate of generation of hole-electron pairs per unit volume (3.1)
G = conductance per unit area of junction
G_n, G_p as for A's
$i = \sqrt{-1}$
I = current density
I_n, I_p = current densities due to electrons and holes (2.5) (2.6) (4.10)
I_{n0}, I_{p0} I_{p1} (4.11) (4.12) (4.18) (4.19)
I_s, I_{ns}, I_{ps} saturation reverse current densities (4.11) (4.18) (4.21)
I_r see text with (4.35)
J = subscript in Section 3 for junction Fig. 5 equation (3.11)
k = Boltzmann's constant
K = space charge parameter (2.17)
L = length
$L_a = n_i/a$ (2.15)
L_D = Debye length (2.12)
L_n, L_p = diffusion lengths for electron in p-region and holes in n-region (4.8)
L_r = length required for potential increase of kT/q in region of constant field (4.32) Appendices II and IV
L_1 corresponds to a-c. field, Appendix IV
n = density of electrons

n_n, n_p = equilibrium densities of electrons in n- and p-regions
p = density of holes
p_n, p_p = equilibrium densities of holes in n- and p-regions
p_0 = d-c. component of non-equilibrium hole density (4.3)
$p_1 \exp(i\omega t)$ = a-c. component of non-equilibrium hole density (4.3)
P = total number per unit area of holes in specimen (2.35)
q = electronic charge ($q = |q|$)
$Q = qP$ = total charge per unit area (2.39)
r = recombination coefficient for holes and electrons (3.1)
R = resistance of unit area
R_0 = resistance of unit area obtained by integrating conductivity (3.10), Appendix I
R_1 = effective series resistance, discussed in connection with (3.13)
s = rate of recombination per unit area of surface per unit hole density, Appendix V
S = susceptance per unit area (imaginary part of admittance)
S_p, S_n, S_T as for A's.
t = time
T = temperature in °K
T = subscript for transition region
$u = q\psi/kT$ (2.9), $q(\psi - \varphi_1)/kT$ (2.32), Appendix VII
v_0 and $v_1 e^{i\omega t}$ = d-c. and a-c. components of voltage applied in forward direction (4.2)
W = width of space charge region in abrupt junction, Section 2.4
w = half thickness of n-region or transistor base of Sections 5 and 6.
w = half width of square rod in Appendix V.
x = coordinate perpendicular to plane of junction
y, z = transverse coordinates, Appendix V
y = reduced length (2.17), Appendix VII
α = current gain factor in transistor (6.4)
α = parameter in Appendix III and VI
α_{ij} = parameter in Appendix V
β_i = parameter in Appendix V
γ = parameter in Appendices II, IV and VII
ϵ = symbol for emitter Section 6
$\theta_i = \beta_i w$ Appendix V
κ = dielectric constant
μ = mobility of a hole ($b\mu$ = mobility of electron)
ν = rates of recombination etc., Appendix VI
ρ = charge density (2.1)
σ = conductivity

σ_i = conductivity of intrinsic material (4.15)
σ_n = conductivity of n-region $\doteq qb\mu n_n$
σ_p = conductivity of p-region $\doteq q\mu p_p$
τ = time
τ_n; τ_p = life times of electrons in p-region and holes in n-region (3.2) (3.3) (4.7)
τ_T = relaxation time of transition region, Appendix IV
$\varphi, \varphi_p, \varphi_n$ = Fermi level and quasi Fermi levels (2.2) (2.4)
$\delta\varphi$ = applied voltage across specimen in forward direction, Section 2.3, (4.2)
χ = sw/D in Appendix V
ψ = electrostatic potential (2.2)
ω = circular frequency of a-c. (4.2)