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Nonequilibrium Quantum Statistics; Application to the Laser*†

VICTOR KORENMAN‡

Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts

A method is presented for describing a general nonequilibrium system in contact with a reservoir in terms of the correlation functions of its quantized field operators. Equations of motion for these correlation functions are derived for a system of multilevel moving atoms interacting with the radiation field, which interacts in turn with the reservoir system. In an appropriate limit these equations are shown to include the usual rate equations for the level distributions. A simple and rigorous description of the influence of a cavity and an optical pump is derived and other types of reservoir coupling are briefly discussed. This description is then applied to a model of a gas laser. The breakdown of the linear theory at the usual lasing threshold suggests consideration of nonlinear terms, which are developed in an expansion in the field strength. Using only the first nonlinear term we find the equations of Lamb by examining the stability of pure modes of the radiation field. An analysis of the incoherent part of the field shows, however, that the existence of a pure mode is precluded by the presence of spontaneous emission of radiation. Nevertheless a value of the linewidth can be plausibly derived from this calculation and is found to be half the Townes value with correction terms. A further discussion of the laser is presented in which the presence of a pure mode is not assumed. It is shown that while a simple incoherent perturbation theory gives divergences, a more self-consistent calculation taking partial coherence into account gives the same results made plausible by pure mode theory.

INTRODUCTION

Despite the many successes of previous descriptions of maser and laser behavior it remains of interest to construct a fully quantum mechanical theory based on the microscopic equations of motion of the various system components, and to show how more phenomenological theories arise from approximations to these equations. In addition, due to the special nature of lasers as nonequilibrium systems whose behavior is determined by the influence of macroscopic reservoirs, certain problems arise in the quantum mechanical discussion whose solution is of

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‡ Present address: Department of Physics and Astronomy, University of Maryland, College Park, Maryland.

intrinsic interest. This paper, then, falls naturally into two main parts. In the first part we present a discussion of nonequilibrium systems in contact with reservoirs, generally applicable but with special emphasis on the interactions relevant to a model of a gas laser. In the appropriate places the connection with previous work on this subject, notably by Schwinger (*1*), Senitzky (*2*), McCumber (*3*), and Feynman and Vernon (*4*) is discussed. In the second part we discuss the laser directly in several approximations. Section II deals with the linear approximation, its breakdown and the stability of a pure oscillation above the breakdown point in a simple nonlinear case corresponding to the calculation of Lamb (*5*). Section III discusses the oscillating nonlinear system further with particular attention to the incoherent part of the electromagnetic field. In Section IV we take account of the impossibility of a true oscillation and discuss the properties of the nonlinear system entirely in terms of correlation functions. The coherence properties (*6*) of the resultant electromagnetic field are touched upon briefly.

I. DESCRIPTION OF NONEQUILIBRIUM SYSTEMS

In any quantum system complex enough to require a statistical description all observables can be expressed as averages of products of small numbers of field operators over the density matrix or ensemble of states appropriate to the system. For example[1] in a system of interacting bosons described by the creation and destruction operators $\psi^\dagger(\mathbf{r}t)$, $\psi(\mathbf{r}t)$ the function $g^<(\mathbf{r}t, \mathbf{r}'t') \equiv -i\langle\psi^\dagger(\mathbf{r}'t')\psi(\mathbf{r}t)\rangle$ has the properties

$$ig^<(\mathbf{r}t, \mathbf{r}t) = \rho(\mathbf{r}t)$$

$$i\int d(\mathbf{r}-\mathbf{r}')e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}g^<(\mathbf{r}t, \mathbf{r}'t) = \rho(\mathbf{k}t)$$

where $\rho(\mathbf{r}t)$ is the density of particles and $\rho(\mathbf{k}t)$ their velocity distribution. The function $g^r(\mathbf{r}t, \mathbf{r}'t') \equiv -i\eta(t-t')\langle[\psi(\mathbf{r}t), \psi^\dagger(\mathbf{r}'t')]\rangle$[2] gives the amplitude for finding a particle at $(\mathbf{r}, t)$ if it was added to the system at $(\mathbf{r}', t')$ earlier. As the description of the time evolution of a disturbance it contains explicitly the energy and decay rate of the disturbance (and thus the single particle excitation spectrum). As the description of a disturbance, however, g^r contains no direct information about the excitation densities in the undisturbed system.

In general functions which depend on two times can be either densities, like $g^<$, or propagators, like g^r. Thus $\langle\psi^\dagger(\mathbf{r}t)\psi(\mathbf{r}t)\psi^\dagger(\mathbf{r}'t')\psi(\mathbf{r}'t')\rangle$ will be related to the intensity of density fluctuations in a system while $\eta(t-t')\langle[\psi^\dagger(\mathbf{r}t)\psi(\mathbf{r}t), \psi^\dagger(\mathbf{r}'t')\psi(\mathbf{r}'t')]\rangle$ will describe the propagation of a single density fluctuation, that

[1] For a fuller discussion of some of the points below see, for example, ref. 7.

[2] $\eta(x) \equiv +1$ for $x > 0$ and 0 for $x < 0$.

is, the frequency and damping of a sound wave. Both types of information are needed for a complete description of the system.

Any calculation of the properties of the system involves coupled equations for the various propagators and densities. In thermal equilibrium there is a great simplification because the excitation densities are completely determined by the spectrum (propagator) and the temperature and chemical potential. Thus

$$ig^{<}(\mathbf{r}\mathbf{r}'\omega) = -2 \,\mathrm{Im}\, g^{r}(\mathbf{r}\mathbf{r}'\omega)(e^{\beta(\omega-\mu)} - 1)^{-1}$$

where β is the inverse temperature in energy units and ω is the variable conjugate to $(t - t')$ under Fourier transformation. We have set $\hbar = 1$.

Using relations such as this, one can write the equations in terms of propagators only. In practice a combination of propagator and density is considered for which the thermal condition becomes a condition of periodicity in imaginary time (*8*, *7*). Solving equations for this combination and invoking the periodicity is equivalent, after analytic continuation back to real time, to the explicit rewriting of densities in terms of propagators and the solution of the resulting propagator equations.

Out of thermal equilibrium there is no *a priori* way of expressing densities in terms of propagators so none of this simplification is possible. It is necessary to consider the full set of coupled equations for these two types of functions. In addition, since the system is specified not by a given density matrix but by initial or boundary conditions or interaction with a reservoir a way must be found to incorporate this information into the determining equations.

In what follows we shall derive equations for the various correlation functions of a nonequilibrium system of interacting atoms and radiation. The procedure we use for deriving these equations is applicable to a much wider class of systems, and the equations derived are amenable to all the approximation techniques[3] which have been developed for systems in equilibrium. In this particular case we find propagator equations which have the same dependence on densities as in equilibrium. The equations for the densities are interpreted as detailed balancing of gains and losses when the system is not changing in time and Boltzmann-like equations when the rate of change is slow. The elimination of reservoir coordinates is straightforward in the cases considered.

A. Hamiltonian and Correlation Functions

Our system consists of a gas of multilevel atoms interacting with the radiation field via electric dipole coupling. The cavity is represented by a reservoir system also coupled to the field. To avoid a proliferation of interactions we think in terms of optical pumping, so that the pump and cavity may be merged into a single system. Correspondingly, atomic collisions will be ignored, though their

[3] See, for example, ref. *9* where some nonperturbative approximations are discussed.

formal inclusion would be straightforward.[4] For definiteness the atomic states will be taken to obey Bose statistics, though Fermi statistics would be equally satisfactory and the quantum nature of the statistics will be irrelevant at the densities of interest.

The Hamiltonian is

$$\begin{aligned} H = \int d\mathbf{r}\{\sum_{\alpha} \psi_\alpha^\dagger(\mathbf{r}t)[\epsilon_\alpha - \nabla^2/2M]\psi_\alpha(\mathbf{r}t) \\ + \tfrac{1}{2}[\mathbf{E}(\mathbf{r}t)\cdot\mathbf{E}(\mathbf{r}t) + c^2(\nabla \times \mathbf{A}(\mathbf{r}t))\cdot(\nabla \times \mathbf{A}(\mathbf{r}t))] \\ - \sum_{\alpha,\beta} A_i(\mathbf{r}t)\gamma^i_{\alpha\beta}\psi_\alpha^\dagger(\mathbf{r}t)\psi_\beta(\mathbf{r}t) - A_i(\mathbf{r}t)J_c^i(\mathbf{r}t) + H_c(\mathbf{r}t)\} \end{aligned} \quad (1.1)$$

where $\psi_\alpha^\dagger(\mathbf{r}t)$ and $\psi_\alpha(\mathbf{r}t)$ are creation and destruction operators for an atom in state α at $\mathbf{r}$ and t and ϵ_α is the energy of this state α. The electric field $\mathbf{E}(\mathbf{r}t)$ is transverse, as is the vector potential $\mathbf{A}(\mathbf{r}t)$. The α, β component of the imaginary antisymmetric matrix γ^i is proportional to the dipole moment in the i direction for the transition between levels α and β. $J_c(\mathbf{r}t)$ and $H_c(\mathbf{r}t)$ are, respectively, the current operator and Hamiltonian density of the reservoir system.[5] The operators obey the usual commutation rules

$$[\psi_\alpha(\mathbf{r}t), \psi_\beta^\dagger(\mathbf{r}'t)] = \delta_{\alpha\beta}\delta(\mathbf{r} - \mathbf{r}') \quad (1.2a)$$

$$[A_i(\mathbf{r}t), E_j(\mathbf{r}'t)] = i(\nabla^2\delta_{ij} - \partial_i\partial_j)(4\pi|\mathbf{r} - \mathbf{r}'|)^{-1} \equiv -i\delta^T_{ij}(\mathbf{r} - \mathbf{r}'). \quad (1.2b)$$

Finally we define a subsidiary Hamiltonian density which will be used to generate equations for correlation functions but which will be set to zero at the end of any calculation.

$$H_\zeta'(\mathbf{r}t) \equiv -\mathbf{j}^\zeta(\mathbf{r}t)\cdot\mathbf{A}(\mathbf{r}t) + \sum_\alpha [\chi^{*\zeta}_\alpha(\mathbf{r}t)\psi_\alpha(\mathbf{r}t) + \psi_\alpha^\dagger(\mathbf{r}t)\chi^\zeta_\alpha(\mathbf{r}t)] \quad (1.3)$$

where $\mathbf{j}^\zeta(\mathbf{r}t)$, $\chi^{*\zeta}_\alpha(\mathbf{r}t)$, and $\chi^\zeta_\alpha(\mathbf{r}t)$ are prescribed numerical functions and $\mathbf{j}^\zeta(\mathbf{r}t)$

[4] The formal inclusion of collisions in the equations for the atomic system is treated in the author's thesis.

[5] One might object that the Hamiltonian (1.1) is not correct since it does not include the term $(e^2/mc^2)\psi^\dagger_\alpha A^2\psi_\alpha$ which normally occurs in the interaction term. Following Fiutak (*10*), however, we can define a canonical transformation which eliminates the A^2 term as far as electric multipoles are concerned. In the dipole approximation the interaction term then has the following properties. As far as the electromagnetic equations of motion are concerned the effective interaction term is $-A(rt)\cdot\partial P(rt)/\partial t$ where $P(rt) = d_{\alpha\beta}\psi^\dagger_\alpha(rt)\psi_\beta(rt)$ and $d_{\alpha\beta}$ is the dipole moment between states α and β. As far as the matter equations of motion are concerned this term is $+P(rt)\cdot\partial A(rt)/\partial t$. The interaction term we use is $-A(rt)\cdot\gamma_{\alpha\beta}\psi^\dagger_\alpha(rt)\psi_\beta(rt)$ where $\gamma_{\alpha\beta} = -i\omega_{\alpha\beta}d_{\alpha\beta}$. It can be shown rigorously that we regain the results of the correct dipole calculation by replacing $\gamma_{\alpha\beta}$ wherever it occurs by $-i\omega d_{\alpha\beta}$ where ω is the frequency of the photon involved in the α, β transition to which $\gamma_{\alpha\beta}$ refers. In order to avoid equations more cumbersome than necessary we use the Hamiltonian as given.

is a transverse vector. The index ζ taken the two values $(+)$ and $(-)$ and its significance will appear shortly.

Our system is described by a density matrix ρ, which is arbitrary except for conditions macroscopically imposed on the system. We work in a Heisenberg picture with respect to the true Hamiltonian. In what follows all times are between t_0 and t_1 where, eventually, $t_1 \to +\infty$. The initial time t_0 is effectively $-\infty$ for a stationary system.

Following Schwinger[6] we define[7, 8]

$$\langle A_i^+(\mathbf{r}t)\rangle \equiv \Delta^{-1} \operatorname{tr}\left\{\left(\exp\left[i\int_{t_0}^{t_1} H'_-(\mathbf{r}'t')\, d\mathbf{r}'\, dt'\right]\right)_- \times \left(\exp\left[-i\int_{t_0}^{t_1} H'_+(\mathbf{r}'t')\, d\mathbf{r}'\, dt'\right] A_i(\mathbf{r}t)\right)_+ \rho\right\} \tag{1.4a}$$

$$\langle A_i^-(\mathbf{r}t)\rangle \equiv \Delta^{-1} \operatorname{tr}\left\{\left(\exp\left[i\int_{t_0}^{t_1} H'_-(\mathbf{r}'t')\, d\mathbf{r}'\, dt'\right] A_i(\mathbf{r}t)\right)_- \times \left(\exp\left[-i\int_{t_0}^{t_1} H'_+(\mathbf{r}'t')\, d\mathbf{r}'\, dt'\right]_+ \rho\right\} \tag{1.4b}$$

with

$$\Delta \equiv \operatorname{tr}\left\{\left(\exp\left[i\int_{t_0}^{t_1} H'_-(\mathbf{r}'t')\, d\mathbf{r}'\, dt'\right]\right)_- \cdot \left(\exp\left[-i\int_{t_0}^{t_1} H'_+(\mathbf{r}'t')\, d\mathbf{r}'\, dt'\right]\right)_+ \rho\right\} \tag{1.4c}$$

The $(+)$ or $(-)$ subscript appended to an exponential integral refers to positive or negative time ordering and $H_\pm'$ is the subsidiary Hamiltonian defined in (1.3). Note that

$$\lim_{H'\to 0} \langle \mathbf{A}^+(\mathbf{r}t)\rangle = \lim_{H'\to 0} \langle \mathbf{A}^-(\mathbf{r}t)\rangle = \operatorname{tr}[\mathbf{A}(\mathbf{r}t)\rho]/\operatorname{tr}[\rho] \equiv \langle \mathbf{A}(\mathbf{r}t)\rangle \tag{1.5}$$

In general the absence of ζ indices on a bracketed quantity will indicate that the limit of vanishing H' has been taken. The convenience of the definition (1.4) is that $\langle \mathbf{A}^+(\mathbf{r}t)\rangle$ and $\langle \mathbf{A}^-(\mathbf{r}t)\rangle$ are generating functionals for the various correla-

[6] See ref. *1*. Our approach and techniques are based on the ideas of this paper.

[7] After this work was complete and some of the results reported (*11*) the similar independent research of L. V. Keldysh (*12*) was brought to our attention. He also treats nonequilibrium systems in terms of the Schwinger generating functional, using diagrammatic techniques.

[8] In certain approximations which make use of a nonlocal subsidiary Hamiltonian in place of (1.3) a somewhat more general form of (1.4) is required. Such a form is employed in the discussion of collisions in the author's thesis.

tion functions of the electromagnetic field operators. We have, for example,

$$\lim_{H'\to 0} \frac{\delta\langle A_i^+(\mathbf{r}t)\rangle}{\delta j_l^+(\mathbf{r}'t')} = i[\langle(A_i(\mathbf{r}t)A_l(\mathbf{r}'t'))_+\rangle - \langle A_i(\mathbf{r}t)\rangle\langle A_l(\mathbf{r}'t')\rangle] \tag{1.6}$$

where the $(+)$ subscript again means positive time ordering. If we define the functions

$$d_{ij}^{>}(\mathbf{r}t, \mathbf{r}'t') \equiv i[\langle A_i(\mathbf{r}t)A_j(\mathbf{r}'t')\rangle - \langle A_i(\mathbf{r}t)\rangle\langle A_j(\mathbf{r}'t')\rangle] \tag{1.7a}$$

$$d_{ij}^{<}(\mathbf{r}t, \mathbf{r}'t') \equiv i[\langle A_j(\mathbf{r}'t')A_i(\mathbf{r}t)\rangle - \langle A_j(\mathbf{r}'t')\rangle\langle A_i(\mathbf{r}t)\rangle] = d_{ji}^{>}(\mathbf{r}'t', \mathbf{r}t) \tag{1.7b}$$

$$d_{ij}^{r}(\mathbf{r}t, \mathbf{r}'t') \equiv \eta(t - t')[d_{ij}^{>}(\mathbf{r}t, \mathbf{r}'t') - d_{ij}^{<}(\mathbf{r}t, \mathbf{r}'t')] \tag{1.7c}$$[2]

$$\begin{aligned} d_{ij}^{r}(\mathbf{r}t, \mathbf{r}'t') &- d_{ij}^{a}(\mathbf{r}t, \mathbf{r}'t') \\ &\equiv d_{ij}^{>}(\mathbf{r}t, \mathbf{r}'t') - d_{ij}^{<}(\mathbf{r}t, \mathbf{r}'t') \end{aligned} \tag{1.7d}$$

we can write the four first functional derivatives of $\langle \mathbf{A}^{\pm}(\mathbf{r}t)\rangle$ as the components of a 2×2 matrix

$$D_{il}^{\zeta\zeta'}(\mathbf{r}t, \mathbf{r}'t') \equiv \delta\langle A_i^{\zeta}(\mathbf{r}t)\rangle/\delta j_l^{\zeta'}(\mathbf{r}'t') \tag{1.8a}$$

where

$$\lim_{H'\to 0} D_{il}^{\zeta\zeta'}(\mathbf{r}t, \mathbf{r}'t') = d_{ij}^{r}(\mathbf{r}t, \mathbf{r}'t')\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix} d_{ij}^{<}(\mathbf{r}t, \mathbf{r}'t') & -d_{ij}^{<}(\mathbf{r}t, \mathbf{r}'t') \\ d_{ij}^{>}(\mathbf{r}t, \mathbf{r}'t') & -d_{ij}^{>}(\mathbf{r}t, \mathbf{r}'t')\end{pmatrix} \tag{1.8b}$$

Since the matrix $\mathbf{D}^{\zeta\zeta'}$ contains both the propagator, d^r, and the photon density (field intensity), $d^<$, it is an appropriate expression to analyze in a discussion of nonequilibrium systems.

In a similar manner we can generate correlation functions for the atomic field operators. Thus, defining

$$\begin{aligned} \langle\psi_\alpha^{(+)}(\mathbf{r}t)\rangle \equiv \Delta^{-1}\operatorname{tr}\Big\{ &\Big(\exp\Big[i\int_{t_0}^{t_1} H'_-(\mathbf{r}'t')\,d\mathbf{r}'\,dt'\Big]\Big)_- \\ &\times \Big(\exp\Big[-i\int_{t_0}^{t_1} H'_+(\mathbf{r}'t')\,d\mathbf{r}'\,dt'\Big]\psi_\alpha(\mathbf{r}t)\Big)_+\rho\Big\} \end{aligned} \tag{1.9}$$

and the analogous expression for $\langle\psi_\alpha^{(-)}(\mathbf{r}t)\rangle$ we have the matrix

$$G_{\alpha\beta}^{\zeta\zeta'}(\mathbf{x}\mathbf{x}') \equiv \delta\langle\psi_\alpha^{\zeta}(\mathbf{x})\rangle/\delta\chi_\beta^{\zeta'}(\mathbf{x}') \tag{1.10a}$$

where

$$\lim_{H'\to 0} G_{\alpha\beta}^{\zeta\zeta'}(\mathbf{x}\mathbf{x}') = g_{\alpha\beta}^{r}(\mathbf{x}\mathbf{x}')\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix} g_{\alpha\beta}^{<}(\mathbf{x}\mathbf{x}') & -g_{\alpha\beta}^{<}(\mathbf{x}\mathbf{x}') \\ g_{\alpha\beta}^{>}(\mathbf{x}\mathbf{x}') & -g_{\alpha\beta}^{>}(\mathbf{x}\mathbf{x}')\end{pmatrix} \tag{1.10b}$$

We have used the abbreviation $(\mathbf{r}t) \equiv (\mathbf{x})$ and defined

$$g^{>}_{\alpha\beta}(\mathbf{xx}') \equiv -i\langle\psi_\alpha(\mathbf{x})\psi_\beta^\dagger(\mathbf{x}')\rangle \tag{1.11a}$$

$$g^{<}_{\alpha\beta}(\mathbf{xx}') \equiv -i\langle\psi_\beta^\dagger(\mathbf{x}')\psi_\alpha(\mathbf{x})\rangle \tag{1.11b}$$

$$g^{r}_{\alpha\beta}(\mathbf{xx}') \equiv \eta(t-t')[g^{>}_{\alpha\beta}(\mathbf{xx}') - g^{<}_{\alpha\beta}(\mathbf{xx}')] \tag{1.11c}$$

$$g^{r}_{\alpha\beta}(\mathbf{xx}') - g^{a}_{\alpha\beta}(\mathbf{xx}') \equiv g^{>}_{\alpha\beta}(\mathbf{xx}') - g^{<}_{\alpha\beta}(\mathbf{xx}') \tag{1.11d}$$

We have also noted that, in the problems under consideration, $\langle\psi_\alpha(\mathbf{r}t)\rangle = 0$, when $H' = 0$.

B. Equations of Motion

We now consider the equations of motion for the generating functionals $\langle A^\pm(\mathbf{x})\rangle$, $\langle\psi_\alpha^\pm(\mathbf{x})\rangle$ and for the correlation functions which we may derive from them. First, for the matter fields, using the Hamiltonian and commutation relations (1.1), (1.2a), (1.3), and (1.9) we find

$$i\partial\langle\psi_\alpha^\zeta(\mathbf{x})\rangle/\partial t = (\epsilon_\alpha - \nabla^2/2M)\langle\psi_\alpha^\zeta(\mathbf{x})\rangle + \chi_\alpha^\zeta(\mathbf{x}) - \sum_\sigma \gamma^i_{\alpha\sigma}\langle A_i^\zeta(\mathbf{x})\psi_\sigma^\zeta(\mathbf{x})\rangle \tag{1.12}$$

Now

$$\delta\langle\psi_\sigma^\zeta(\mathbf{x})\rangle/\delta j_i^{\zeta'}(\mathbf{x}') = i\zeta[\langle A_i^{\zeta'}(\mathbf{x}')\psi_\sigma^\zeta(\mathbf{x})\rangle - \langle A_i^{\zeta'}(\mathbf{x}')\rangle\langle\psi_\sigma^\zeta(\mathbf{x})\rangle] \tag{1.13}$$

(the factor ζ reflects the different sign in the argument of the exponential integral of H_+' and H_-' in (1.9)). Then we may write

$$(i\partial/\partial t - \epsilon_\alpha + \nabla^2/2M)\langle\psi_\alpha^\zeta(\mathbf{x})\rangle + \gamma^i_{\alpha\sigma}(\langle A_i^\zeta(\mathbf{x})\rangle - i\zeta\delta/\delta j_i^\zeta(\mathbf{x}))\langle\psi_\sigma^\zeta(\mathbf{x})\rangle = \chi_\alpha^\zeta(\mathbf{x}) \tag{1.12$'$}$$

where we now use the summation convention for repeated state and polarization indices. Taking functional derivatives of (1.12′) with respect to χ and then setting $\chi = 0$ we find equations of motion for the various correlation functions. Thus, using the definition (1.10) of $\mathbf{G}^{\zeta\zeta'}$ we have

$$\left(i\frac{\partial}{\partial t} - \epsilon_\alpha + \frac{\nabla^2}{2M}\right)G^{\zeta\zeta'}_{\alpha\beta}(\mathbf{xx}') + \gamma^i_{\alpha\sigma}\left(\langle A_i^\zeta(\mathbf{x})\rangle - i\zeta\frac{\delta}{\delta j_i^\zeta(\mathbf{x})}\right)G^{\zeta\zeta'}_{\sigma\beta}(\mathbf{xx}') = \delta_{\alpha\beta}\delta^{\zeta\zeta'}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') \equiv \delta^{\zeta\zeta'}_{\alpha\beta}(\mathbf{x}-\mathbf{x}') \tag{1.14}$$

We now assume that the matrix $\mathbf{G}$ has an inverse.[9] That is, we can define a

[9] This assumption effectively limits us to systems which have been set up long enough ago that initial atomic correlations have died out. For a discussion of the initial value problem a different procedure must be used. This point is discussed in more detail in the author's thesis. In any case (1.15) is applicable to the laser, which is a steady state system.

function $\mathbf{G}^{-1}$ such that

$$\sum_{\zeta_1} \int d\mathbf{r}_1\, dt_1\, G^{-1}\,{}^{\zeta\zeta_1}_{\alpha\sigma}(\mathbf{r}t, \mathbf{r}_1t_1) G^{\zeta_1\zeta'}_{\sigma\beta}(\mathbf{r}_1t_1, \mathbf{r}'t') = \delta^{\zeta\zeta'}_{\alpha\beta}(\mathbf{r} - \mathbf{r}')\delta(t' - t') \tag{1.15}$$

We use the relation

$$\delta\mathbf{G}/\delta j = -\mathbf{G}(\delta\mathbf{G}^{-1}/\delta j)\mathbf{G} \tag{1.16}$$

to rewrite the equation for $\mathbf{G}$ as

$$\begin{aligned}(i\partial/\partial t - \epsilon_\alpha + \nabla^2/2M)G^{\zeta\zeta'}_{\alpha\beta}(\mathbf{x}\mathbf{x}') + \gamma^i_{\alpha\sigma}\langle A_i^\zeta(\mathbf{x})\rangle G^{\zeta\zeta'}_{\sigma\beta}(\mathbf{x}\mathbf{x}') \\ + i\zeta\gamma^i_{\alpha\sigma} \int d\mathbf{x}_1\, d\mathbf{x}_2\, G^{\zeta\zeta_1}_{\sigma\lambda}(\mathbf{x}\mathbf{x}_1) \frac{\delta G^{-1}\,{}^{\zeta_1\zeta_2}_{\lambda\tau}(\mathbf{x}_1\mathbf{x}_2)}{\delta j_i^\zeta(\mathbf{x})} G^{\zeta_2\zeta'}_{\tau\beta}(\mathbf{x}_2\mathbf{x}') \\ = \delta^{\zeta\zeta'}_{\alpha\beta}(\mathbf{x} - \mathbf{x}')\end{aligned} \tag{1.17}$$

Defining the "mass operator"

$$\begin{aligned}M^{\zeta\zeta_1}_{\alpha\sigma}(\mathbf{x}\mathbf{x}_1) \equiv &- \gamma^i_{\alpha\sigma}\langle A_i^\zeta(\mathbf{x})\rangle \delta^{\zeta\zeta_1}(\mathbf{x} - \mathbf{x}_1) \\ &- i\zeta\gamma^i_{\alpha\rho} \int d\mathbf{x}_2 G^{\zeta\zeta_2}_{\rho\lambda}(\mathbf{x}\mathbf{x}_2)\delta G^{-1}\,{}^{\zeta_2\zeta_1}_{\lambda\sigma}(\mathbf{x}_2\mathbf{x}_1)/\delta j_i^\zeta(\mathbf{x})\end{aligned} \tag{1.18}$$

we can write

$$\begin{aligned}\int d\mathbf{x}_1 \left[\left(i\frac{\partial}{\partial t} - \epsilon_\alpha + \frac{\nabla^2}{2M}\right)\delta^{\zeta\zeta_1}_{\alpha\sigma}(\mathbf{x} - \mathbf{x}_1) - M^{\zeta\zeta_1}_{\alpha\sigma}(\mathbf{x}\mathbf{x}_1)\right] \\ \cdot G^{\zeta_1\zeta'}_{\sigma\beta}(\mathbf{x}_1\mathbf{x}') = \delta^{\zeta\zeta'}_{\alpha\beta}(\mathbf{x} - \mathbf{x}')\end{aligned} \tag{1.19}$$

This identifies the quantity in square brackets as $G^{-1}\,{}^{\zeta\zeta_1}_{\alpha\sigma}(\mathbf{x}\mathbf{x}_1)$ and allows us to rewrite (1.18) as

$$\begin{aligned}M^{\zeta\zeta_1}_{\alpha\sigma}(\mathbf{x}\mathbf{x}_1) = &- \gamma^i_{\alpha\sigma}\langle A_i^\zeta(\mathbf{x})\rangle \delta^{\zeta\zeta_1}(\mathbf{x} - \mathbf{x}_1) \\ &+ i\zeta\gamma^i_{\alpha\rho} \int d\mathbf{x}_2 G^{\zeta\zeta_2}_{\rho\lambda}(\mathbf{x}\mathbf{x}_2)\delta M^{\zeta_2\zeta_1}_{\lambda\sigma}(\mathbf{x}_2\mathbf{x}_1)/\delta j_i^\zeta(\mathbf{x})\end{aligned} \tag{1.18$'$}$$

The iterative solution of (1.18′) to terms in γ^2 is

$$M^{\zeta\zeta_1}_{\alpha\sigma}(\mathbf{x}\mathbf{x}_1) \approx -\gamma^i_{\alpha\sigma}\langle A_i^\zeta(\mathbf{x})\rangle\delta^{\zeta\zeta_1}(\mathbf{x} - \mathbf{x}_1) - i\zeta\gamma^i_{\alpha\rho}G^{\zeta\zeta_1}_{\rho\lambda}(\mathbf{x}\mathbf{x}_1)\gamma^j_{\lambda\sigma}D^{\zeta_1\zeta}_{ji}(\mathbf{x}_1\mathbf{x}) \tag{1.20}$$

If $\langle\mathbf{A}\rangle \to 0$ when $\mathbf{j} \to 0$, as is the case in a nonoscillating system we may write

$$M^{\zeta\zeta_1}_{\alpha\sigma}(\mathbf{x}\mathbf{x}_1) \approx i\gamma^i_{\alpha\rho}\gamma^j_{\sigma\lambda}G^{\zeta\zeta_1}_{\rho\lambda}(\mathbf{x}\mathbf{x}_1)D^{\zeta\zeta_1}_{ij}(\mathbf{x}\mathbf{x}_1)\zeta_1 \tag{1.20$'$}$$

where we have used the antisymmetry of $\gamma^i_{\alpha\beta}$ and the relation which is easily demonstrated from (1.7) (1.8)

$$D^{\zeta\zeta'}_{ij}(\mathbf{x}\mathbf{x}')\zeta' = D^{\zeta'\zeta}_{ji}(\mathbf{x}'\mathbf{x})\zeta \tag{1.21}$$

A particular evaluation of $\mathbf{M}$ is tantamount to an evaluation of $\mathbf{G}$ and thus $\mathbf{g}^r$ and $\mathbf{g}^<$. As we shall shortly show this first approximation (1.20′) to $\mathbf{M}$ will lead to a propagator with the usual "golden rule" value for the atomic level widths and an equation for $\mathbf{g}^<$ which is a generalized rate equation for the populations of these levels. First, however, we must decompose the matrix equation (1.19) into equations for the components $\mathbf{g}^r$ and $\mathbf{g}^<$.

Calling

$$(i\partial/\partial t - \epsilon_\alpha + \nabla^2/2M)\delta_{\alpha\sigma}(\mathbf{x} - \mathbf{x}_1) \equiv g^0_{\alpha\sigma}(\mathbf{x}\mathbf{x}_1)^{-1} \tag{1.22}$$

we may write (1.19) as

$$\int d\mathbf{x}_1 \left\{ g^0_{\alpha\sigma}(\mathbf{x}\mathbf{x}_1)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} m^{++}_{\alpha\sigma}(\mathbf{x}\mathbf{x}_1) & m^{+-}_{\alpha\sigma}(\mathbf{x}\mathbf{x}_1) \\ m^{-+}_{\alpha\sigma}(\mathbf{x}\mathbf{x}_1) & m^{--}_{\alpha\sigma}(\mathbf{x}\mathbf{x}_1) \end{pmatrix} \right\}$$
$$\times \left\{ g^r_{\sigma\beta}(\mathbf{x}_1\mathbf{x}') \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} g^<_{\sigma\beta}(\mathbf{x}_1\mathbf{x}') & -g^<_{\sigma\beta}(\mathbf{x}_1\mathbf{x}') \\ g^>_{\sigma\beta}(\mathbf{x}_1\mathbf{x}') & -g^>_{\sigma\beta}(\mathbf{x}_1\mathbf{x}') \end{pmatrix} \right\} = \delta_{\alpha\beta}(\mathbf{x} - \mathbf{x}') \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.23}$$

It is easy to show[10] that this leads to the equations[11]

$$\int d\mathbf{x}_1 [g^0_{\alpha\sigma}(\mathbf{x}\mathbf{x}_1)^{-1} - m^r_{\alpha\sigma}(\mathbf{x}\mathbf{x}_1)] g^r_{\sigma\beta}(\mathbf{x}_1\mathbf{x}') = \delta_{\alpha\beta}(\mathbf{x} - \mathbf{x}') \tag{1.24a}$$

$$\int d\mathbf{x}_1 [g^a_{\alpha\sigma}(\mathbf{x}\mathbf{x}_1)^{-1} - m^a_{\alpha\sigma}(\mathbf{x}\mathbf{x}_1)] g^a_{\sigma\beta}(\mathbf{x}_1\mathbf{x}') = \delta_{\alpha\beta}(\mathbf{x} - \mathbf{x}') \tag{1.24b}$$

$$\int d\mathbf{x}_1 [g^0_{\alpha\sigma}(\mathbf{x}\mathbf{x}_1)^{-1} - m^r_{\alpha\sigma}(\mathbf{x}\mathbf{x}_1)] g^{>(<)}_{\sigma\beta}(\mathbf{x}_1\mathbf{x}') = \int d\mathbf{x}_1 m^{>(<)}_{\alpha\lambda}(\mathbf{x}\mathbf{x}_1) g^a_{\lambda\beta}(\mathbf{x}_1\mathbf{x}') \tag{1.24c}$$

where $\mathbf{M}$ is rewritten in the form (cf. 1.10b)

$$M_{\alpha\beta}(\mathbf{x}\mathbf{x}') = m^r_{\alpha\beta}(\mathbf{x}\mathbf{x}') \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} m^<_{\alpha\beta}(\mathbf{x}\mathbf{x}') & -m^<_{\alpha\beta}(\mathbf{x}\mathbf{x}') \\ m^>_{\alpha\beta}(\mathbf{x}\mathbf{x}') & -m^>_{\alpha\beta}(\mathbf{x}\mathbf{x}') \end{pmatrix} \tag{1.25a}$$

and

$$m^r_{\alpha\beta}(\mathbf{x}\mathbf{x}') - m^a_{\alpha\beta}(\mathbf{x}\mathbf{x}') = m^>_{\alpha\beta}(\mathbf{x}\mathbf{x}') - m^<_{\alpha\beta}(\mathbf{x}\mathbf{x}') \tag{1.25b}$$

Note that if $\mathbf{m}^r$ is a retarded function we also have, in analogy to (1.11c),

$$m^r_{\alpha\beta}(\mathbf{x}\mathbf{x}') = \eta(t - t')[m^>_{\alpha\beta}(\mathbf{x}\mathbf{x}') - m^<_{\alpha\beta}(\mathbf{x}\mathbf{x}')] \tag{1.25c}$$

This will not always be the case.

[10] Summing the top row of (1.23) we find, schematically, $((g^0)^{-1} - m^{++} - m^{+-})g^r = 1$. Similarly, the sum of the bottom row is $((g^0)^{-1} - m^{--} - m^{-+})g^r = 1$. Assuming that g^r has a unique inverse we have $m^{++} + m^{+-} = m^{--} + m^{-+} \equiv m^r$ and then (1.24a). Defining $m^{+-} \equiv -m^<$ and $m^{-+} \equiv m^>$ we are led to (1.25a). If we define m^a by (1.25b) and use (1.11d) we can find (1.24b) and (1.24c) by considering other components of (1.23).

[11] Kadanoff and Baym (7), Chap. 8, have derived Eqs. (1.24) for an interacting particle system driven away from equilibrium by an external force. They do not consider the more general case of nonthermal equilibrium being maintained through contact with a reservoir.

C. Significance of Atomic Equations of Motion

In order to make contact with conventional discussions of interacting atoms and radiation we will analyze Eqs. (1.24) further. For simplicity we consider a system with translational invariance in space as well as in time so that Fourier transforms in both space- and time-difference variables are appropriate. That is

$$g^{<}_{\alpha\beta}(\mathbf{r}t, \mathbf{r}'t') \equiv \int \frac{d^3\mathbf{k}d\omega}{(2\pi)^4} g^{<}_{\alpha\beta}(\mathbf{k}\omega)e^{i[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]} \tag{1.26}$$

and similar definitions hold for all other relevant functions. Other simplifications will be made as they become useful. Then, using (1.22), (1.24a) becomes

$$[(\omega - \epsilon_\alpha - k^2/2M)\delta_{\alpha\sigma} - m^r_{\alpha\sigma}(\mathbf{k}\omega)]g^r_{\sigma\beta}(\mathbf{k}\omega) = \delta_{\alpha\beta} \tag{1.27}$$

In most cases it is a good approximation to ignore the off diagonal parts of $m^r_{\alpha\sigma}(\mathbf{k}\omega)$ when computing the diagonal part of $g^r_{\alpha\beta}$. In that case we find

$$g^r_{\alpha\alpha}(\mathbf{k}\omega) \approx [\omega - \epsilon_\alpha - k^2/2M - m^r_{\alpha\alpha}(\mathbf{k}\omega)]^{-1} \tag{1.27$'$}$$

When there is translational invariance in time it is possible to show that $m^r_{\alpha\alpha}(\mathbf{k}\omega)$ is analytic in the upper half ω plane and has the representation

$$m^r_{\alpha\alpha}(\mathbf{k}\omega) = m^1_{\alpha\alpha}(\mathbf{k}\omega) - \tfrac{1}{2}i\bar{m}_{\alpha\alpha}(\mathbf{k}\omega) \tag{1.28a}$$

where

$$m^1_{\alpha\alpha}(\mathbf{k}\omega) = P\int \frac{d\omega'}{2\pi}\frac{\bar{m}_{\alpha\alpha}(\mathbf{k}\omega')}{\omega - \omega'} \tag{1.28b}$$

$\mathbf{m}^1$ and $\bar{\mathbf{m}}$ are both real and $\omega\bar{m}_{\alpha\alpha}(\mathbf{k}\omega)$ is positive.

If $m^1_{\alpha\alpha}(\mathbf{k}\omega)$ is slowly varying as a function of ω we can define $E(\mathbf{k})$ such that

$$E_\alpha(\mathbf{k}) = \epsilon_\alpha + k^2/2M + m^1\alpha\alpha(\mathbf{k}, E_\alpha(\mathbf{k})) \tag{1.29}$$

If $\bar{m}_{\alpha\alpha}(\mathbf{k}\omega)$ is also slowly varying near $\omega = E_\alpha(\mathbf{k})$ we may to a good approximation write (1.27′) as

$$g^r_{\alpha\alpha}(\mathbf{k}\omega) \approx [\omega - E_\alpha(\mathbf{k}) + \tfrac{1}{2}i\bar{m}_{\alpha\alpha}(\mathbf{k}, E_\alpha(\mathbf{k}))]^{-1} \tag{1.30}$$

and then

$$g^r_{\alpha\alpha}(\mathbf{k}, t - t') \approx -i\eta(t - t')e^{-iE_\alpha(\mathbf{k})(t-t')}e^{-(1/2)\bar{m}_{\alpha\alpha}(\mathbf{k}, E_\alpha(\mathbf{k}))(t-t')} \tag{1.30$'$}$$

Thus $\tfrac{1}{2}\bar{m}_{\alpha\alpha}(\mathbf{k}, E_\alpha(\mathbf{k}))$ is the decay rate of the amplitude for finding an atom in state α with momentum $\mathbf{k}$ at time t when it was introduced into the system at t' earlier. Then $\bar{m}_{\alpha\alpha}(\mathbf{k}, E_\alpha(\mathbf{k}))$ is the corresponding rate of atomic state decay, the inverse lifetime. $E_\alpha(\mathbf{k})$ is the energy of the atom in state α with momentum $\mathbf{k}$. From (1.29) we see that this includes an energy of interaction, $m^1_{\alpha\alpha}(\mathbf{k}, E_\alpha(\mathbf{k}))$, along with the unperturbed energy and the kinetic term.

When any of the assumptions which we made above are not valid the form

(1.30′) of $\mathbf{g}^r(t - t')$ will be somewhat modified, often not exhibiting pure exponential decay. In any case, however, $\bar{\mathbf{m}}$ will be closely related to a decay rate and $\mathbf{m}^1$ to an energy of interaction.

Let us see what our first approximation (1.20′) gives for the decay rate. Using (1.8b), (1.10b), (1.25a) we find

$$m_{\alpha\alpha}^{>(<)}(\mathbf{x}\mathbf{x}_1) \approx i\gamma_{\alpha\lambda}^{i}\gamma_{\alpha\sigma}^{j}g_{\lambda\sigma}^{>(<)}(\mathbf{x}\mathbf{x}_1)\, d_{ij}^{>(<)}(\mathbf{x}\mathbf{x}_1) \tag{1.31}$$

In this approximation we take $g_{\lambda\sigma}^{>(<)}(\mathbf{x}\mathbf{x}_1)$ to be diagonal in λ, σ and $d_{ij}^{>(<)}(\mathbf{x}\mathbf{x}_1)$ diagonal in ij. Then, remembering that $\gamma_{\alpha\lambda}^{i}$ is pure imaginary

$$m_{\alpha\alpha}^{>(<)}(\mathbf{k}\omega) \approx -|\gamma_{\alpha\sigma}^{i}|^2 i \int \frac{d^3\mathbf{k}_1\, d\omega_1}{(2\pi)^4}\, g_{\sigma\sigma}^{>(<)}(\mathbf{k}_1\omega_1)\, d_{ii}^{>(<)}(\mathbf{k} - \mathbf{k}_1, \omega - \omega_1) \tag{1.31′}$$

It is easy to verify from (1.20′) that (1.25c) holds whereupon

$$\begin{aligned}\bar{m}_{\alpha\alpha}(\mathbf{k}\omega) &= |\gamma_{\alpha\sigma}^{i}|^2 \int \frac{d^3\mathbf{k}_1\, d\omega_1}{(2\pi)^4} \\ &\cdot[g_{\sigma\sigma}^{>}(\mathbf{k}_1\omega_1)\, d_{ii}^{>}(\mathbf{k} - \mathbf{k}_1, \omega - \omega_1) - g_{\sigma\sigma}^{<}(\mathbf{k}_1\omega_1) d_{ii}^{<}(\mathbf{k} - \mathbf{k}_1, \omega - \omega_1)]\end{aligned} \tag{1.32}$$

Now in thermal equilibrium it is easy to show (7) that

$$g_{\alpha\beta}^{>}(\mathbf{k}\omega) = e^{\beta(\omega-\mu)} g_{\alpha\beta}^{<}(\mathbf{k}\omega) \tag{1.33}$$

where β is the inverse temperature and μ the chemical potential. Then, using (1.11d)

$$\begin{aligned} g_{\alpha\beta}^{<}(\mathbf{k}\omega) &= g_{\alpha\beta}^{<}(\mathbf{k}\omega)[g_{\alpha\beta}^{>}(\mathbf{k}\omega) - g_{\alpha\beta}^{<}(\mathbf{k}\omega)]^{-1}[g_{\alpha\beta}^{r}(\mathbf{k}\omega) - g_{\alpha\beta}^{a}(\mathbf{k}\omega)] \\ &= (e^{\beta(\omega-\mu)} - 1)^{-1}[g_{\alpha\beta}^{r}(\mathbf{k}\omega) - g_{\alpha\beta}^{a}(\mathbf{k}\omega)] \end{aligned} \tag{1.33′}$$

Similarly, the relation

$$d_{ij}^{>}(\mathbf{k}\omega) = e^{\beta\omega}\, d_{ij}^{<}(\mathbf{k}\omega) \tag{1.34}$$

leads to

$$d_{ij}^{<}(\mathbf{k}\omega) = (e^{\beta\omega} - 1)^{-1}[d_{ij}^{r}(\mathbf{k}\omega) - d_{ij}^{a}(\mathbf{k}\omega)] \tag{1.34′}$$

In our static but nonthermal case we can define dimensionless functions, corresponding to the thermal factors above, by the relations

$$g_{\alpha\beta}^{<}(\mathbf{k}\omega) \equiv n_{\alpha\beta}(\mathbf{k}\omega)[g_{\alpha\beta}^{r}(\mathbf{k}\omega) - g_{\alpha\beta}^{a}(\mathbf{k}\omega)] \tag{1.35a}$$

$$d_{ij}^{<}(\mathbf{k}\omega) \equiv \mathfrak{N}_{ij}(\mathbf{k}\omega)[d_{ij}^{r}(\mathbf{k}\omega) - d_{ij}^{a}(\mathbf{k}\omega)] \tag{1.35b}$$

From (1.11d) and (1.7d) we then also have

$$g_{\alpha\beta}^{>}(\mathbf{k}\omega) = (1 + n_{\alpha\beta}(\mathbf{k}\omega))[g_{\alpha\beta}^{r}(\mathbf{k}\omega) - g_{\alpha\beta}^{a}(\mathbf{k}\omega)] \tag{1.36a}$$

$$d_{ij}^{>}(\mathbf{k}\omega) = (1 + \mathfrak{N}_{ij}(\mathbf{k}\omega))[d_{ij}^{r}(\mathbf{k}\omega) - d_{ij}^{a}(\mathbf{k}\omega)] \tag{1.36b}$$

Using the definition (1.11) we can show

$$g^a_{\alpha\beta}(\mathbf{k}\omega) = g^r_{\beta\alpha}(\mathbf{k}\omega)^* \tag{1.37}$$

so that, from (1.27) and (1.28) (note no sum)

$$\begin{aligned} g^r_{\alpha\alpha}(\mathbf{k}\omega) - g^a_{\alpha\alpha}(\mathbf{k}\omega) &\equiv -i\bar{g}_{\alpha\alpha}(\mathbf{k}\omega) \\ &= \frac{-i\bar{m}_{\alpha\alpha}(\mathbf{k}\omega)}{[\omega - \epsilon_\alpha - k^2/2M - m^1_{\alpha\alpha}(\mathbf{k}\omega)]^2 + [\tfrac{1}{2}\bar{m}_{\alpha\alpha}(\mathbf{k}\omega)]^2}. \end{aligned} \tag{1.38}$$

Similarly (see below)

$$\begin{aligned} d^r_{ii}(\mathbf{k}\omega) - d^a_{ii}(\mathbf{k}\omega) &\equiv i\bar{d}_{ii}(\mathbf{k}\omega) \\ &= \frac{i\omega\bar{p}_{ii}(\mathbf{k}\omega)}{[\omega^2 - c^2k^2 + p^1_{ii}(\mathbf{k}\omega)]^2 + [\tfrac{1}{2}\omega\bar{p}_{ii}(\mathbf{k}\omega)]^2}. \end{aligned} \tag{1.39}$$

We have introduced the energy spectrum $\bar{g}_{\alpha\alpha}(\mathbf{k}\omega)$ for atoms in state α with momentum $\mathbf{k}$ and $\bar{d}_{ii}(\mathbf{k}\omega)$ for photons polarized in direction i with momentum $\mathbf{k}$. From the commutation relations (1.2) and definitions of $\mathbf{g}^r$, $\mathbf{g}^a$, $\mathbf{d}^r$, and $\mathbf{d}^a$ we find the normalization conditions

$$\int \frac{d\omega}{2\pi}\, \bar{g}_{\alpha\alpha}(\mathbf{k}\omega) = 1 \tag{1.40}$$

$$\int \frac{d\omega}{2\pi}\, \omega\bar{d}_{ii}(\mathbf{k}\omega) = 1 - k_i^2/k^2 \text{ [12]} \tag{1.41}$$

With these definitions (1.32) becomes

$$\begin{aligned} \bar{m}_{\alpha\alpha}(\mathbf{k}\omega) = |\gamma^i_{\alpha\sigma}|^2 \int \frac{d^3\mathbf{k}_1\, d\omega_1}{(2\pi)^4}\, \bar{g}_{\sigma\sigma}(\mathbf{k}_1\omega_1)\bar{d}_{ii}(\mathbf{k} - \mathbf{k}_1, \omega - \omega_1) \\ \times \{(1 + n_{\sigma\sigma}(\mathbf{k}_1\omega_1))(1 + \mathfrak{N}_{ii}(\mathbf{k} - \mathbf{k}_1, \omega - \omega_1)) \\ - n_{\sigma\sigma}(\mathbf{k}_1\omega_1)\mathfrak{N}_{ii}(\mathbf{k} - \mathbf{k}_1, \omega - \omega_1)\} \end{aligned} \tag{1.42}$$

In the limit $\bar{\mathbf{m}} \to 0$, $\bar{\mathbf{p}} \to 0$, where we ignore the lifetimes of the photons and atoms, $\bar{g}_{\alpha\alpha}(\mathbf{k}\omega)$ and $\bar{d}_{ii}(\mathbf{k}\omega)$ become δ functions and we can perform some of the integrations in (1.42). We find

$$\begin{aligned} \bar{m}_{\alpha\alpha}(\mathbf{k}\omega) = |\gamma^i_{\alpha\sigma}|^2 \int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\, \frac{1 - (k - k_1)_i^2/|\mathbf{k} - \mathbf{k}_1|^2}{2c'|\mathbf{k} - \mathbf{k}_1|} \\ \times \{\delta(\omega - E_\sigma(\mathbf{k}_1) - c'|\mathbf{k} - \mathbf{k}_1|)[(1 + n_\sigma(\mathbf{k}_1))(1 + \mathfrak{N}_i(\mathbf{k} - \mathbf{k}_1)) \\ - n_\sigma(\mathbf{k}_1)\mathfrak{N}_i(\mathbf{k} - \mathbf{k}_1)] + \delta(\omega - E_\sigma(\mathbf{k}_1) + c'|\mathbf{k} - \mathbf{k}_1|) \\ \cdot[(1 + n_\sigma(\mathbf{k}_1))\mathfrak{N}_i(\mathbf{k}_1 - \mathbf{k}) - n_\sigma(\mathbf{k}_1)(1 + \mathfrak{N}_i(\mathbf{k}_1 - \mathbf{k}))]\} \end{aligned} \tag{1.42'}$$

[12] The factor $1 - k_i^2/k^2$ reflects the transverse nature of the radiation fields.

where we have used the true state energy (1.29) and the true velocity of light in the medium, c'. We have defined

$$n_{\sigma\sigma}(\mathbf{k}, E_\sigma(\mathbf{k})) \equiv n_\sigma(\mathbf{k}) \tag{1.43}$$

$$\mathfrak{N}_{ii}(\mathbf{k}, c' \mid \mathbf{k} \mid) \equiv \mathfrak{N}_i(\mathbf{k}) \tag{1.44}$$

and have used the relation

$$1 + \mathfrak{N}_{ij}(\mathbf{k}\omega) = -\mathfrak{N}_{ji}(-\mathbf{k}, -\omega) \tag{1.45}$$

which follows from the definitions (1.35), (1.7).

We are only interested in $\bar{m}_{\alpha\alpha}(\mathbf{k}\omega)$ for values of ω near $\omega = E_\alpha(\mathbf{k})$. Then the first δ function picks out those states $(\sigma, \mathbf{k}_1)$ which lie below state $(\alpha, \mathbf{k})$ in energy and those photons which can be emitted by state $(\alpha, \mathbf{k})$ in a transition to $(\sigma, \mathbf{k}_1)$ conserving both energy and momentum. $| \gamma^i_{\alpha\sigma} |^2$ is the squared matrix element and $(1 + n_\sigma)(1 + \mathfrak{N}_i)$ the correct density of states for this transition taking account of the Bose statistics of both atoms and photons. The subtracted term, $n_\sigma \mathfrak{N}_i$, is the correct density of states to represent the rate at which an atom in $(\alpha, \mathbf{k})$ *induces* atoms in $(\sigma, \mathbf{k}_1)$ to absorb a photon and jump into $(\alpha, \mathbf{k})$. The difference between these two terms is then the *net* decay rate of an atom in $(\alpha, \mathbf{k})$ to $(\sigma, \mathbf{k}_1)$. The induced term, which is present due to the Bose nature of the atoms, is negligible at the densities of interest where $n_\sigma(\mathbf{k})$ is always very small compared to one.

The second δ function picks out states which lie above $(\alpha, \mathbf{k})$. The two terms give the difference between the rate at which an atom in $(\alpha, \mathbf{k})$ absorbs a photon and jumps up to $(\sigma, \mathbf{k}_1)$ and the rate at which it induces an atom in $(\sigma, \mathbf{k}_1)$ to emit a photon and drop to $(\alpha, \mathbf{k})$. This difference is the net rate at which an atom in (α, k) will decay by transitions to higher energy states.

Then our expression (1.42′) for $\bar{m}_{\alpha\alpha}(\mathbf{k}\omega)$ is easily interpretable in terms of the first order transitions taking atoms out of the state $(\alpha, \mathbf{k})$ and is equivalent to the usual "golden rule" calculation. If we had considered the form (1.42) we would have been describing the same processes with suitable account being taken of both photon lifetimes and atomic level widths. By continuing our expansion, (1.20), of $\mathbf{M}$ to higher order or producing a better, nonperturbative calculation of $\mathbf{M}$ we could find a more accurate value for the decay rate. Since the interaction energy, $m^1_{\alpha\alpha}(\mathbf{k}, \omega)$ is just the Hilbert transform of $\bar{m}_{\alpha\alpha}(\mathbf{k}\omega)$, (1.28), our calculation of this energy must also reproduce the known results in the appropriate limits.

Now let us examine the equation for $\mathbf{g}^<$. If we multiply (1.24c) on the left by $\mathbf{g}^r$, take Fourier transforms, and assume that only diagonal terms are important we find

$$g^<_{\alpha\alpha}(\mathbf{k}\omega) = g^r_{\alpha\alpha}(\mathbf{k}\omega) m^<_{\alpha\alpha}(\mathbf{k}\omega) g^a_{\alpha\alpha}(\mathbf{k}\omega) \tag{1.46}$$

which is, considering (1.27′), (1.28), and (1.37)

$$g^{<}_{\alpha\alpha}(\mathbf{k}\omega) = \frac{m^{<}_{\alpha\alpha}(\mathbf{k}\omega)}{[\omega - \epsilon_\alpha - k^2/2M - m^1_{\alpha\alpha}(\mathbf{k}\omega)]^2 + [\frac{1}{2}\bar{m}_{\alpha\alpha}(\mathbf{k}\omega)]^2} \tag{1.46'}$$

or, using (1.38),

$$ig^{<}_{\alpha\alpha}(\mathbf{k}\omega)\bar{m}_{\alpha\alpha}(\mathbf{k}\omega) = \bar{g}_{\alpha\alpha}(\mathbf{k}\omega)im^{<}_{\alpha\alpha}(\mathbf{k}\omega) \tag{1.46''}$$

Now $(2\pi)^{-1}\int d\omega\, ig^{<}_{\alpha\alpha}(\mathbf{k}\omega)$ is, by (1.11b), the number of particles in a unit cell of phase space about $\mathbf{k}$. Then $ig^{<}_{\alpha\alpha}(\mathbf{k}\omega)$ is the density of particles in state $(\alpha, \mathbf{k})$ per unit energy interval around ω. With $\bar{m}_{\alpha\alpha}(\mathbf{k}\omega)$ the net decay rate per particle in $(\alpha, \mathbf{k})$ at energy ω, the left hand side of (1.46″) is the net rate at which particles leave $(\alpha, \mathbf{k})$ per unit energy interval. We claim that the right hand side of this equation is the rate of spontaneous transition of atoms into $(\alpha, \mathbf{k})$ per unit ω. Using (1.31′) with (1.35) and (1.38) we have

$$\begin{aligned} i\bar{g}_{\alpha\alpha}(\mathbf{k}\omega)m^{<}_{\alpha\alpha}(\mathbf{k}\omega) \approx \bar{g}_{\alpha\alpha}(\mathbf{k}\omega)\,|\gamma^{i}_{\alpha\sigma}|^2 \int \frac{d^3\mathbf{k}_1\,d\omega_1}{(2\pi)^4} \\ \cdot\bar{g}_{\sigma\sigma}(\mathbf{k}_1\omega_1)\bar{d}_{ii}(\mathbf{k}-\mathbf{k}_1, \omega-\omega_1) \times n_{\sigma\sigma}(\mathbf{k}_1\omega_1)[\eta(\omega-\omega_1)\mathfrak{N}_{ii}(\mathbf{k}-\mathbf{k}_1, \omega-\omega_1) \\ + \eta(\omega_1-\omega)(1+\mathfrak{N}_{ii}(\mathbf{k}_1-\mathbf{k}, \omega_1-\omega))] \end{aligned} \tag{1.47}$$

This is indeed the rate of transition from $(\sigma, \mathbf{k}_1, \omega_1)$ to $(\alpha, \mathbf{k}, \omega)$ with the absorption or emission of a single photon. It is the rate of *spontaneous* transitions since the final state factor $\bar{g}_{\alpha\alpha}(\mathbf{k}\omega)$ is multiplied by unity rather than $(1 + n_{\alpha\alpha}(\mathbf{k}\omega))$ which is appropriate for the total, spontaneous plus induced, transition rate.

Then our first approximation to $\mathbf{M}$ leads to a generalized rate equation, balancing gain and loss, for particle state occupation densities taking account of all first order processes. We assert that this will remain the case for any approximation to M, higher approximations merely taking account of other processes. Again, when any of the approximations we have made to get to (1.46″) are not valid, (1.24c) will still determine the particle densities with $m^{<}_{\alpha\beta}(\mathbf{xx}')$ a spontaneous emission rate in a suitably generalized sense. In particular, if the system is not in a steady state but is undergoing slow time variation, one can repeat the treatment of Kadanoff and Baym[13] who show that the equation for $\mathbf{g}^{<}$ is, in that case, equivalent to the usual Boltzmann equation for the particle densities.

If we knew the photon densities $\mathfrak{N}_{ii}(\mathbf{k}\omega)$ and the spectral function $\bar{d}_{ii}(\mathbf{k}\omega)$ we could, in principle, use (1.46) and (1.47) to solve for the particle densities $ig^{<}_{\alpha\alpha}(\mathbf{k}\omega)$. In a realistic system, however, the photon correlation functions are not specified and we must solve coupled equations for the electromagnetic and particle excitations. We now proceed to sketch the derivation of the equations for the photon field.

[13] See ref. 7, Chap. 9.

D. Equation of Motion for Radiation Field

From the Hamiltonian (1.1), (1.3) and commutation relations (1.2) we find the equation of motion

$$(\partial^2/\partial t^2 - c^2\nabla^2)\langle A_i^\zeta(\mathbf{r}t)\rangle \equiv \Box\langle A_i^\zeta(\mathbf{r}t)\rangle = j_i^\zeta(\mathbf{r}t) + \langle J_i^\zeta(\mathbf{r}t)\rangle \qquad (1.48)$$

where we have combined the two currents

$$\sum_{\alpha,\beta} \gamma_{\alpha\beta}^i \psi_\alpha^\dagger(\mathbf{r}t)\psi_\beta(\mathbf{r}t) + J_c^i(\mathbf{r}t) \equiv J_i(\mathbf{r}t) \qquad (1.49)$$

Equations for the various electromagnetic correlation functions follow by taking functional derivatives of (1.48) with respect to **j**. Thus, using (1.8)

$$\Box D_{ij}^{\zeta\zeta'}(\mathbf{x}\mathbf{x}') = \delta_{ij}^{\zeta\zeta'}(\mathbf{x} - \mathbf{x}') + \delta\langle J_i^\zeta(\mathbf{x})\rangle/\delta j_j^{\zeta'}(\mathbf{x}') \qquad (1.50)$$

One useful way to treat this equation is to perform a change of variables, expressing $\langle J_i^\zeta(\mathbf{x})\rangle$ as a functional of $\langle \mathbf{A}\rangle$ rather than **j**.[14] Then we may rewrite (1.50) as

$$\int d\mathbf{x}_1(\Box\delta_{il}^{\zeta\zeta_1}(\mathbf{x} - \mathbf{x}_1) - P_{il}^{\zeta\zeta_1}(\mathbf{x}\mathbf{x}_1))D_{lj}^{\zeta_1\zeta'}(\mathbf{x}_1\mathbf{x}') = \delta_{ij}^{\zeta\zeta'}(\mathbf{x} - \mathbf{x}') \qquad (1.51)$$

where

$$P_{il}^{\zeta\zeta_1}(\mathbf{x}\mathbf{x}_1) \equiv \delta\langle J_i^\zeta(\mathbf{x})\rangle/\delta\langle A_l^{\zeta_1}(\mathbf{x}_1)\rangle \qquad (1.52)$$

Using arguments similar to those leading to (1.24) and (1.25) it is easy to show that P is of the form

$$\mathbf{P}_{ij}(\mathbf{x}\mathbf{x}') = p_{ij}^r(\mathbf{x}\mathbf{x}')\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}p_{ij}^<(\mathbf{x}\mathbf{x}') & -p_{ij}^<(\mathbf{x}\mathbf{x}')\\ p_{ij}^>(\mathbf{x}\mathbf{x}') & -p_{ij}^>(\mathbf{x}\mathbf{x}')\end{pmatrix} \qquad (1.53a)$$

$$p_{ij}^r(\mathbf{x}\mathbf{x}') - p_{ij}^a(\mathbf{x}\mathbf{x}') = p_{ij}^>(\mathbf{x}\mathbf{x}') - p_{ij}^<(\mathbf{x}\mathbf{x}') \qquad (1.53b)$$

while

$$\int d\mathbf{x}_1[\Box\delta_{il}(\mathbf{x} - \mathbf{x}_1) - p_{il}^r(\mathbf{x}\mathbf{x}_1)]\, d_{lj}^r(\mathbf{x}_1\mathbf{x}') = \delta_{ij}(\mathbf{x} - \mathbf{x}') \qquad (1.54a)$$

$$\int d\mathbf{x}_1[\Box\delta_{il}(\mathbf{x} - \mathbf{x}_1) - p_{il}^a(\mathbf{x}\mathbf{x}_1)]\, d_{lj}^a(\mathbf{x}_1\mathbf{x}') = \delta_{ij}(\mathbf{x} - \mathbf{x}') \qquad (1.54b)$$

$$\int d\mathbf{x}_1[\Box\delta_{il}(\mathbf{x} - \mathbf{x}_1) - p_{il}^r(\mathbf{x}\mathbf{x}_1)]\, d_{lj}^{>(<)}(\mathbf{x}_1\mathbf{x}') = \int d\mathbf{x}_1\, p_{ik}^{>(<)}(\mathbf{x}\mathbf{x}_1)\, d_{kj}^a(\mathbf{x}_1\mathbf{x}')^{15} \qquad (1.54c)$$

[14] Such a change of variables is used in a similar context in ref. *13*.

[15] Equations similar in appearance to (1.54) are derived in refs. *1*, *2*, and *3*, but there are important differences between our equations and theirs. First of all (1.54) are exact while the previous results derive from a factorization of correlation functions. Correspondingly our evaluations of the dielectric functions p^r, $p^<$ differ from those of the previous authors. Further, the near coherence which is a striking characteristic of the laser field is present in our work but absent from theirs.

When we can assert that the inverse of the retarded function $d^r_{ij}(\mathbf{xx}')$ is itself a retarded function we have, further,

$$p^r_{ij}(\mathbf{xx}') = \eta(t - t')[p^>_{ij}(\mathbf{xx}') - p^<_{ij}(\mathbf{xx}')] \tag{1.53c}$$

To make contact with conventional treatments of electromagnetic radiation we again consider a fully translationally invariant system where (1.54a) becomes

$$d^r_{ii}(\mathbf{k}\omega) = -[\omega^2 - c^2k^2 + p^1_{ii}(\mathbf{k}\omega) + \tfrac{1}{2}i\omega\bar{p}_{ii}(\mathbf{k}\omega)]^{-1} \tag{1.55}$$

and, in view of (1.53c),

$$p^1_{ii}(\mathbf{k}\omega) = P\int \frac{d\omega'}{2\pi}\,\frac{\omega'\bar{p}_{ii}(\mathbf{k}\omega')}{\omega' - \omega} \tag{1.56}$$

When $p^1_{ii}(\mathbf{k}\omega)$ is slowly varying we can define c' such that

$$c^2k^2 - p^1_{ii}(\mathbf{k}, c'\,|\,\mathbf{k}\,|) = c'^2k^2 \tag{1.57}$$

and, when $\omega\bar{p}(\mathbf{k}\omega)$ is small and slowly varying near $\omega = c'\,|\,\mathbf{k}\,|$, we find

$$d^r_{ii}(\mathbf{k}, t - t') \approx \eta(t - t')(1/c'\,|\,\mathbf{k}\,|)\sin[c'\,|\,\mathbf{k}\,|(t - t')]e^{-\bar{p}_{ii}(\mathbf{k},c'|\mathbf{k}|)(t-t')/4} \tag{1.55'}$$

An electromagnetic disturbance then propagates with velocity c', giving an effective index of refraction $n^2(k\omega) = 1 + p^1(\mathbf{k}\omega)/\omega^2$. The field decays with constant $\tfrac{1}{4}\bar{p}(\mathbf{k}\omega)$ so that the energy density, or intensity, decays with constant $\tfrac{1}{2}\bar{p}(\mathbf{k}\omega)$. The relation (1.56) reduces to the usual Kramers-Kronig relation (*14*) between the real and imaginary parts of the susceptibility. Again, when the approximations above are not valid, (1.54a) gives a correct description of the propagation of an electromagnetic field disturbance, whether or not susceptibility and exponential decay are relevant concepts in this description.

From the formal similarity between the equations (1.54) for **D** and those (1.24) for **G** we will expect that $-ip^<_{ii}(\mathbf{k}\omega)/2\omega$ will give the rate of spontaneous emission of photons into $(\mathbf{k}, \omega)$ with polarization i. In conjunction with the identification of $\tfrac{1}{2}\bar{p}_{ii}(\mathbf{k}\omega)$ as the decay rate and the relation (1.53c, 1.54a, 1.55)

$$i\omega\bar{p}_{ii}(\mathbf{k}\omega) = p^>_{ii}(\mathbf{k}\omega) - p^<_{ii}(\mathbf{k}\omega) \tag{1.58}$$

this allows us the interpretation of the net decay of a single photon being the difference between absorption $-ip^>_{ii}(\mathbf{k}\omega)/2\omega$ and the emission it induces $-ip^<_{ii}(\mathbf{k}\omega)/2\omega$, with the spontaneous emission rate automatically equal to the rate of stimulated emission due to a single photon. In thermal equilibrium we have

$$p^>(\mathbf{k}\omega) = e^{\beta\omega}p^<(\mathbf{k}\omega) \qquad \text{(cf. 1.34)} \tag{1.59}$$

which reproduces the well known relationship between absorption and stimulated emission in that case. A first perturbation calculation of **P** would bear out these interpretations as did the lowest order calculation of **M**. We defer such a calculation to the next section.

E. Influence of Reservoir

We have so far considered the equations of motion for the matter and field correlation functions but have not, as yet, had occasion to specify the statistical state of or boundary conditions on our system. This becomes necessary when we look more closely at our expression for **P**, (1.52). Using (1.49) we can write

$$P_{ij}^{\zeta\zeta'}(\mathbf{x}\mathbf{x}') = \delta/\delta\langle A_j^{\zeta'}(\mathbf{x}')\rangle\{i\zeta\gamma_{\alpha\beta}^{i}G_{\beta\alpha}^{\zeta\zeta}(\mathbf{x}\mathbf{x}^{\zeta}) + \langle J_c^{i\zeta}(\mathbf{x})\rangle\} \tag{1.60}$$

where $\mathbf{x}^{\zeta} \equiv \mathbf{r}t^{+}$ or $\mathbf{r}t^{-}$ for $\zeta = (+), (-)$ respectively. Using (1.19) this is

$$P_{ij}^{\zeta\zeta'}(\mathbf{x}\mathbf{x}') = i\zeta\gamma_{\alpha\beta}^{i}\int d\mathbf{x}_1\, d\mathbf{x}_2\, G_{\beta\delta}^{\zeta\zeta_1}(\mathbf{x}\mathbf{x}_1)\frac{\delta M_{\delta\lambda}^{\zeta_1\zeta_2}(\mathbf{x}_1\mathbf{x}_2)}{\delta\langle A_j^{\zeta'}(x')\rangle}G_{\lambda\alpha}^{\zeta_2\zeta'}(\mathbf{x}_2\mathbf{x}') + P_{cij}^{\zeta\zeta'}(\mathbf{x}\mathbf{x}') \tag{1.60'}$$

The first term is the contribution of the active atoms and is entirely expressible in terms of their properties and correlation functions of the electromagnetic field. The second contains, in $p_{cij}^{r}(\mathbf{x}\mathbf{x}')$ and $p_{cij}^{<}(\mathbf{x}\mathbf{x}')$, the damping and energy shift of electromagnetic field fluctuations due to interaction with the reservoir as well as the spontaneous emission of electromagnetic energy into the region of interest by the reservoir system. In a system such as the laser, where we may consider the power spectrum of the pump lamp as known, along with the conductivity, shape, and temperature of the resonant cavity, it is clear that we can consider $\mathbf{p}_c^{r}(\mathbf{x}\mathbf{x}')$ and $\mathbf{p}_c^{<}(\mathbf{x}\mathbf{x}')$ and thus $P_{cij}^{\zeta\zeta'}(\mathbf{x}\mathbf{x}')$ as known functions.[16] In fact we must specify this function in order to specify the nature of the particular system under investigation, and this is completely equivalent to the specification of the density operator ρ. That is to say, we have found a means of implicitly specifying the density matrix of the system in terms of the physical statements about the system which actually define it.

Although this type of implicit specification of the density matrix is particularly simple for the reservoir interaction treated here it is by no means limited to this case. Thus, in our discussion of the equations of motion for **G**, a knowledge of the

[16] Compare Feynman and Vernon (*4*). Their equation (4.2) corresponds to our use of $\bar{p}_c(xx_1)$ as an impedance, and the statement that this is the response to a classical force is related to our Eq. (1.60'), $P_c(xx') = \delta\langle J_c(x)\rangle/\delta\langle A(x')\rangle$ as $\langle A(x)\rangle$ is a classical field. The discussion after their Eq. (4.42) is the thermal equilibrium equivalent of our use of $p_c^{<}$ in an equation such as (1.54c) as the rate of spontaneous and thermal emission into the system from the reservoir. The advantages of our formulation are that we are not restricted, as they are, to systems which are essentially interacting oscillators, and that we simultaneously present, along with a representation of the influence of a reservoir, a means for computing the properties of the system of interest in the presence of this influence. In refs. *1*, *2*, and *3* the complete polarization operator, p^r, $p^<$ is to be specified.

photon propagation characteristics and densities would have determined the atomic level densities and thus the statistical state of the system of atoms considered to be in the presence of a "photon reservoir." This is a more complex situation because the reservoir coordinates occur in the mass operator in combination with system coordinates not, as in the simple case above, in a single separate term. The interaction is correspondingly only treated in perturbation theory or some other approximation as opposed to the exact treatment above. It is also possible, at least formally, to treat the case where the reservoir interaction is collision of the active atoms with the cavity walls or other atoms with a given velocity and state distribution.[4] Detailed calculation of the properties of the real He-Ne laser (*15*) in this way, however, would be quite difficult, since the He and Ne atoms as well as the exciting electrons and the electromagnetic field would all have to be treated as interacting dynamical systems.

II. APPLICATION TO LASER

We shall now apply the formalism developed in the previous section to a more detailed analysis of our gas laser model. First we specify the model in some more detail.

In order to avoid the difficulties associated with the consideration of a realistic cavity, we think of our system of atoms, field, and reservoir as filling all space, or more previsely, filling a large volume with periodic boundary conditions so that a traveling wave Fourier decomposition of all the fields is appropriate. The reservoir medium is given specified dielectric properties, which suffices to determine the statistical properties of the system. In particular the resonant nature of the cavity is reproduced by insisting that the medium be strongly absorbing except for a certain discrete set of wavelengths of the electromagnetic field. The absorption will be nonisotropic so we may, and shall, insist that all resonant modes have the same direction of propagation. In fact it will be convenient in part of what follows to specify a lack of inversion symmetry so that running waves, but not standing waves, correspond to cavity modes. We do not yet make that specification. The effect of an optical pump is included by making $\mathbf{p}_c^<(\mathbf{k}\omega)$ large at the pumping frequency. Since $\mathbf{p}_c^<(\mathbf{k}\omega) = (e^{\beta\omega} - 1)^{-1} i\omega\bar{\mathbf{p}}_c(\mathbf{k}\omega)$ in equilibrium (see (1.58), (1.59)) and $\bar{\mathbf{p}}_c(\mathbf{k}\omega)$ is fixed by the cavity absorption at this frequency, this corresponds to giving the cavity a wave-number dependent effective temperature which is large in the pumping region.

For any particular type of excitation mechanism the generalized rate equations (e.g., (1.24c) or (1.46″), (1.47)) can, in principle, be solved to yield the atomic level and velocity distribution. In practice, however, this is quite difficult. One can argue that for most pumping systems the velocity distribution of the atoms is not strongly changed. We will then, for simplicity, take the atomic distribution under the influence of the pump but ignoring the laser signal itself to be Gaussian in the velocities with some nonthermal distribution among the levels. We further allow this level distribution to have a weak space dependence to make allowance for spatially nonuniform pumping. We do not allow it to be time dependent as we are thinking of a steady pump and therefore a laser whose characteristics do not change in time. The laser signal itself will have further effects on these distributions which will be specifically treated in what follows.

Below threshold then we may write

$$g^{<}_{\alpha\beta}(\mathbf{r}t, \mathbf{r}'t') \equiv \int \frac{d^3\mathbf{k}\, d\omega\, d^3\mathbf{K}}{(2\pi)^7} g^{<}_{\alpha\beta}(\mathbf{k}, \omega, \mathbf{K}) \cdot \exp\{i[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') - \omega(t-t') + \tfrac{1}{2}\mathbf{K}\cdot(\mathbf{r}+\mathbf{r}')]\} \tag{2.1}$$

As in (1.35) we define

$$g^{<}_{\alpha\beta}(\mathbf{k}, \omega, \mathbf{K}) \equiv n_{\alpha\beta}(\mathbf{k}, \omega, \mathbf{K})[g^{r}_{\alpha\beta}(\mathbf{k}, \omega, \mathbf{K}) - g^{a}_{\alpha\beta}(\mathbf{k}, \omega, \mathbf{K})] \tag{2.2}$$

Now $\mathbf{g}^r$ is not strongly dependent on the environment, since the major contribution to the retarded mass operator is spontaneous emission (see (1.42')). We then expect it to be a good approximation to ignore the $\mathbf{K}$ dependence of $\mathbf{g}^r$ and $\mathbf{g}^a$. Further, for a given $\mathbf{k}$, $[\mathbf{g}^r(\mathbf{k}\omega) - \mathbf{g}^a(\mathbf{k}\omega)]$ is only appreciable for a very small range of ω (see (1.38)), small compared to the range over which $n(\omega)$ is expected to vary. Then we may set

$$n_{\alpha\beta}(\mathbf{k}, \omega, \mathbf{K}) = n_{\alpha\beta}(\mathbf{k}, \omega_{\alpha\beta}(\mathbf{k}), \mathbf{K})$$

and ignore the ω dependence of n entirely. Finally, anticipating that the off-diagonal density is small we write

$$g^{<}_{\alpha\beta}(\mathbf{k}, \omega, \mathbf{K}) \approx n_{\alpha\alpha}(\mathbf{k}, \mathbf{K})[g^{r}_{\alpha\alpha}(\mathbf{k}\omega) - g^{a}_{\alpha\alpha}(\mathbf{k}\omega)]\delta_{\alpha\beta} \tag{2.2'}$$

where, according to our assumption about the velocity distribution[17]

$$n_{\alpha\alpha}(\mathbf{k}, \mathbf{K}) \equiv n_{\alpha}(\mathbf{K})(2\pi\beta/M)^{3/2} \exp(-\beta k^2/2M) \tag{2.3}$$

In order to satisfy (1.11d) we have

$$g^{>}_{\alpha\beta}(\mathbf{k}, \omega, \mathbf{K}) \approx (1 + n_{\alpha\alpha}(\mathbf{k}, \mathbf{K}))[g^{r}_{\alpha\alpha}(\mathbf{k}\omega) - g^{a}_{\alpha\alpha}(\mathbf{k}\omega)]\delta_{\alpha\beta} \tag{2.4}$$

and we will usually neglect $n_{\alpha\alpha}(\mathbf{k}, \mathbf{K})$ relative to 1. The parameters which remain in discussing the pump are now only the numbers $n_{\alpha}(\mathbf{K})$. We will henceforth specify the pump entirely by specifying these numbers.

[17] Although we always use the same symbol β^{-1} for effective temperature we stress that the temperature in (2.3) need have no relation to the effective temperature of the cavity wall.

A. LINEAR THEORY AND LASER INSTABILITY

We now go on to study the dielectric properties of the nonoscillating system. From (1.60′) the active-atom contribution to the polarization operator **P** is

$$\begin{aligned} P^{\zeta\zeta'}_{Mij}(\mathbf{xx}') &\equiv P^{\zeta\zeta'}_{ij}(\mathbf{xx}') - P^{\zeta\zeta'}_{cij}(\mathbf{xx}') \\ &= i\zeta\gamma^i_{\alpha\beta} G^{\zeta\zeta_1}_{\beta\delta}(\mathbf{xx}_1)\,\frac{\delta M^{\zeta_1\zeta_2}_{\delta\lambda}(\mathbf{x}_1\mathbf{x}_2)}{\delta\langle A^{\zeta'}_j(\mathbf{x}')\rangle}\,G^{\zeta_2\zeta}_{\lambda\alpha}(\mathbf{x}_2\mathbf{x}) \end{aligned} \tag{2.5}$$

From (1.20) the leading contribution to (2.5) is

$$P^{\zeta\zeta'}_{Mij}(\mathbf{xx}') \approx -i\zeta\gamma^i_{\alpha\beta}G^{\zeta\zeta'}_{\beta\delta}(\mathbf{xx}')\gamma^j_{\delta\lambda}G^{\zeta'\zeta}_{\lambda\alpha}(\mathbf{x}'\mathbf{x}) \tag{2.6}$$

This is in the canonical form (1.53) with

$$p^{\gtrless}_{Mij}(\mathbf{xx}') \approx -i\gamma^i_{\alpha\beta}\gamma^j_{\delta\lambda}g^{\gtrless}_{\beta\delta}(\mathbf{xx}')g^{\lessgtr}_{\lambda\alpha}(\mathbf{x}'\mathbf{x}) \tag{2.7}$$

The retarded part (1.53c) can be written

$$p^r_{Mij}(\mathbf{xx}') = -i\gamma^i_{\alpha\beta}\gamma^j_{\delta\lambda}[g^r_{\beta\delta}(\mathbf{xx}')g^<_{\lambda\alpha}(\mathbf{x}'\mathbf{x}) - g^<_{\beta\delta}(\mathbf{xx}')g^a_{\lambda\alpha}(\mathbf{x}'\mathbf{x})]$$

or, using the simplified form (2.2′)

$$\begin{aligned} p^r_{Mij}(\mathbf{k},\omega,\mathbf{K}) \approx i\gamma^i_{\alpha\beta}\gamma^j_{\beta\alpha}\int\frac{d^3\mathbf{k}_1\,d\omega_1}{(2\pi)^4}\,g^r_{\beta\beta}(\mathbf{k}+\mathbf{k}_1,\omega+\omega_1)g^a_{\alpha\alpha}(\mathbf{k}_1\omega_1) \\ \times[n_{\alpha\alpha}(\mathbf{k}_1,\mathbf{K}) - n_{\beta\beta}(\mathbf{k}+\mathbf{k}_1,\mathbf{K})] \end{aligned} \tag{2.8}$$

where we have noted that $\mathbf{g}^r(\mathbf{xx}')\mathbf{g}^r(\mathbf{x}'\mathbf{x})$ vanishes identically. The **K** dependence is not of interest here so we ignore it temporarily. We assume that we may represent the particle propagator in terms of a simple exponential decay law (see 1.30)

$$g^r_{\alpha\alpha}(\mathbf{k}\omega) = [\omega - \epsilon_\alpha - k^2/2M + i\gamma_\alpha]^{-1} \tag{2.9}$$

where ϵ_α now contains the energy of interaction with the medium and γ_α is the inverse level lifetime. Then we rewrite (2.8) as

$$\begin{aligned} p^r_{Mij}(\mathbf{k},\omega) &= i\gamma^i_{\alpha\beta}\gamma^j_{\beta\alpha}\int\frac{d^3\mathbf{k}_1\,d\omega_1}{(2\pi)^4} \\ &\quad\cdot\frac{n_{\alpha\alpha}(\mathbf{k}_1) - n_{\beta\beta}(\mathbf{k}+\mathbf{k}_1)}{(\omega+\omega_1-\epsilon_\beta+(k+k_1)^2/2M+i\gamma_\beta)(\omega_1-\epsilon_\alpha-k_1^2/2M-i\gamma)_\alpha)} \\ &= -\gamma^i_{\alpha\beta}\gamma^j_{\beta\alpha}\int\frac{d^3\mathbf{k}_1}{(2\pi)^3} \\ &\quad\cdot\frac{n_{\alpha\alpha}(\mathbf{k}_1) - n_{\beta\beta}(\mathbf{k}+\mathbf{k}_1)}{\omega-(\epsilon_\beta-\epsilon_\alpha)-k^2/2M-\mathbf{k}\cdot\mathbf{k}_1/M+i(\gamma_\alpha+\gamma_\beta)} \end{aligned} \tag{2.10}$$

Calling

$$\epsilon_\beta - \epsilon_\alpha \equiv \omega_{\beta\alpha} \qquad \gamma_\alpha + \gamma_\beta \equiv \gamma_{\alpha\beta} \tag{2.11}$$

and using (2.3) this is

$$p^r_{Mij}(\mathbf{k}, \omega) = \gamma^i_{\alpha\beta}\gamma^j_{\beta\alpha}(\pi^{1/2}/c|\mathbf{k}|)(\tfrac{1}{2}\beta Mc^2)^{1/2}$$
$$\times \left\{ n_\alpha f\left[(\tfrac{1}{2}\beta Mc^2)^{1/2} \frac{\omega - \omega_{\beta\alpha} - k^2/2M + i\gamma_{\alpha\beta}}{c|\mathbf{k}|} \right] - n_\beta f\left[(\tfrac{1}{2}\beta Mc^2)^{1/2} \frac{\omega - \omega_{\beta\alpha} + k^2/2M + i\gamma_{\alpha\beta}}{c|\mathbf{k}|} \right] \right\} \tag{2.12}$$

where

$$f(z) \equiv \int_{-\infty}^{\infty} \frac{dy}{\pi} \frac{\exp(-y^2)}{y - z} \tag{2.13a}$$

We note the following properties of $f(z)$.[18]

$$f(z) = \pm i \exp(-z^2) - 2\pi^{-1/2} \int_0^z \exp(y^2 - z^2)\, dy \qquad \mathrm{Im}\, z \gtrless 0 \tag{2.13b}$$

$$f(z) \sim -1/\pi^{1/2} z, \qquad |z| \text{ large} \tag{2.13c}$$

Now $(\tfrac{1}{2}\beta Mc^2)^{1/2}\gamma_{\alpha\beta}/c|\mathbf{k}|$ is the ratio of the natural linewidth for the $\alpha - \beta$ transition to the Doppler width, which ratio we shall assume to be small. Noting further that for the wave numbers of interest $k^2/2M$ can be neglected relative to $\omega_{\beta\alpha}$ we may write, using (2.13),

$$p^r_{Mij}(\mathbf{k}, \omega) = \gamma^i_{\alpha\beta}\gamma^j_{\beta\alpha}(\pi^{1/2}/c|\mathbf{k}|)(\tfrac{1}{2}\beta Mc^2)^{1/2}(n_\alpha - n_\beta) \times [i \exp(-z^2_{\alpha\beta}) - 2\pi^{-1/2} f_1(z_{\alpha\beta})] \tag{2.14}$$

where

$$z_{\alpha\beta} \equiv (\tfrac{1}{2}\beta Mc^2)^{1/2}(\omega - \omega_{\beta\alpha})/c|\mathbf{k}|$$

and

$$f_1(x) \equiv \int_0^x \exp(y^2 - x^2)\, dy \tag{2.15}$$

As in (1.55′) the imaginary part of (2.14) is the lowest order contribution of the active atoms to photon decay, and we have found the usual absorption spectrum of a Doppler broadened line. The real part gives the associated contribution to the dielectric constant, and is related to the absorption by a Kramers-Kronig relation. We note that other limits than that we have chosen are included in the more general form (2.12). Thus, when the natural linewidth is large com-

[18] These can be simply obtained by differentiating (2.13a) with respect to z and integrating the resulting differential equation.

pared to the Doppler width or we are concerned with frequencies far from the line center, the asymptotic evaluation of $f(z)$, (2.13c), reproduces the expected Lorentzian line shape.

For future reference we note that (2.7) can be evaluated as

$$\begin{aligned} p^{>}_{Mij}(\mathbf{k}\omega) &= 2i\gamma^{i}_{\alpha\beta}\gamma^{j}_{\beta\alpha}(\pi^{1/2}/c\,|\,\mathbf{k}\,|)(\tfrac{1}{2}\beta Mc^{2})^{1/2}n_{\alpha}\exp(-z^{2}_{\alpha\beta}) \\ p^{<}_{Mij}(\mathbf{k}\omega) &= 2i\gamma^{i}_{\alpha\beta}\gamma^{j}_{\beta\alpha}(\pi^{1/2}/c\,|\,\mathbf{k}\,|)(\tfrac{1}{2}\beta Mc^{2})^{1/2}n_{\beta}\exp(-z^{2}_{\alpha\beta}) \end{aligned} \tag{2.16}$$

For ω positive and close to a transition energy $\omega_{\gamma\delta}$, note that the dominant term of $\mathbf{p}^{<}(\omega)$ is proportional to the number of atoms in the upper transition level while the dominant term of $\mathbf{p}^{>}$ involves the atoms in the lower level. Thus $p^{<}$ describes spontaneous and stimulated emission while $p^{>}$ describes absorption, in agreement with the general discussion following (1.58).

Now that we have evaluated the active atom contribution to the polarization operator in lowest order we turn to an analysis of the propagator $\mathbf{d}^{r}$ which describes the behavior of an electromagnetic excitation in the system.

We first make a remark about the matrix character of $\mathbf{P}_{Mij}$ in the polarization indices i, j. The existence of a dipole transition element $\gamma^{i}_{\alpha\beta}$ between two levels, α and β, in a system with rotational invariance implies that at least one of the two levels is degenerate. If we suppose that one level is a p state, we may take as those linear combinations which we label β, γ, etc. those which transform, respectively, like the x, y, z components of a vector. If α is an s state, the existence of $\gamma^{i}_{\alpha\beta}$ will imply that $\gamma^{j}_{\alpha\beta} = 0$ for $j \neq i$. This is saying no more than that an electromagnetic field scattered from an atom will normally not have its polarization changed. The argument also holds when the levels possess higher symmetries. Then to the order we have computed $\mathbf{P}_{Mij}$ is diagonal. We take $\mathbf{P}_{cij}$ diagonal as well.

From (1.54a) we write, using (2.14), (2.5)

$$\begin{aligned} -[d^{r}_{ij}(\mathbf{k}\omega)]^{-1} &= \omega^{2} - c^{2}k^{2} + p^{r}_{cij}(\mathbf{k}\omega) \\ &\qquad + |\,\gamma^{i}_{\alpha\beta}\,|^{2}\alpha\pi^{1/2}\,|\,c\mathbf{k}\,|^{-1}(n_{\alpha} - n_{\beta}) \\ &\quad \cdot\{i\exp(-\alpha^{2}(\omega - \omega_{\beta\alpha})^{2}/c^{2}k^{2}) - 2\pi^{-1/2}f_{1}[\alpha(\omega - \omega_{\beta\alpha})/c\,|\,\mathbf{k}\,|]\} \end{aligned} \tag{2.17}$$

where

$$\alpha \equiv (\tfrac{1}{2}\beta Mc^{2})^{1/2} \approx 10^{6} \qquad \text{for our system.} \tag{2.18}$$

Now we will only see interesting effects of the medium when the density dependent term is comparable in size to the cavity term, and this will only be the case where the cavity term is itself small. Then we limit our analysis to those values of $\mathbf{k}$ where $p_{c}^{r}(\mathbf{k}\omega)$ is small, that is, those values of $\mathbf{k}$ corresponding to cavity modes.

If we write

$$p^{r}_{cii}(\mathbf{k}\omega) = p^{1}_{cii}(\mathbf{k}\omega) + \tfrac{1}{2}i\omega\bar{p}_{cii}(\mathbf{k}\omega) \tag{2.19}$$

where $\mathbf{p}_c^1$ and $\bar{\mathbf{p}}_c$ are real, the resonant frequency ω_0' of the empty cavity is determined by the equation (see (1.55), (1.57))

$$\omega_0'^2 - c^2k^2 + p_{cii}^1(\mathbf{k}, \omega_0') = 0 \tag{2.20a}$$

while the electromagnetic energy density in an empty cavity decays like $\exp[-\tfrac{1}{2}\bar{p}_c(\mathbf{k}\omega_0')t]$ which identifies

$$\bar{p}_c(\mathbf{k}\omega_0') = 2\omega_0'/Q_{\mathbf{k}} \tag{2.20b}$$

where $Q_{\mathbf{k}}$ is the "quality factor" of the mode and is on the order of 10^8 for a Fabry-Perot cavity. We henceforth assume that the cavity line is Lorentzian and thus ignore any frequency dependence of $p_c^r(\mathbf{k}\omega)$, so that the cavity dielectric characteristics for a given value of $\mathbf{k}$ are completely described by specifying ω_0' and $\bar{p}_c(\mathbf{k}, \omega_0')$ or, equivalently, ω_0' and $Q_{\mathbf{k}}$.

Due to the rapid falloff of the Gaussian and f_1 in (2.14) the only large contribution from the active atoms will be from pairs of levels whose energy separation $\omega_{\beta\alpha}$ is close to ω_0'. We suppose there is one such pair (plus degenerate states). The contribution of other pairs of levels will be a slowly varying function of the frequency and can be included in $\mathbf{p}_c^r$. We relabel the pair of levels of interest by $\alpha, \beta = 1, 2$ where $\epsilon_2 > \epsilon_1$; $\omega_{21} > 0$. Taking ω positive and close to ω_0' we write $\omega_0 + \omega_0' \approx 2\omega_0'$, $c|\mathbf{k}| \approx \omega_0'$ and, using (2.20), we write (2.17) as

$$\begin{aligned} 2\omega_0' d_{ii}^r(\mathbf{k}\omega) = -[\omega - \omega_0' + \tfrac{1}{4}i\bar{p}_c(\mathbf{k}, \omega_0') \\ + (\alpha\pi^{1/2}|\gamma_{12}^i|^2/2\omega_0'^2)(n_1 - n_2)\{i\exp(-\alpha^2(\omega - \omega_{21})^2/\omega_0'^2) \\ - 2\pi^{-1/2}f_1[\alpha(\omega - \omega_{21})/\omega_0']\}]^{-1} \end{aligned} \tag{2.21}$$

It would be easy to analyze (2.21) further to find the change of line center, width, and shape due to the active atoms. Of principal interest, however, is the fact that when the level population structure is normal, $n_1 > n_2$, the added decay term is positive. When there is a population inversion the cavity line is narrowed by the introduction of the active atoms. Such a situation is characteristic of an amplifier. A field excitation introduced into the medium would extract energy from the atoms only to lose it to the cavity. If a large part of the cavity "loss" were due to transmission out of the system, the physical result could be amplification of an incoming signal. We will not stop to discuss the amplifier further but move on to the subject of major interest by noting that if we allow $n_2 - n_1$ to increase sufficiently the expression (2.21) for the propagator becomes physically untenable.

First, to simplify writing, we define some convenient dimensionless variables and parameters

$$\alpha(\omega - \omega_{21})/\omega_0' \equiv z \qquad \alpha(\omega_0' - \omega_{21})/\omega_0' \equiv z_0' \tag{2.22a}$$

$$\alpha^2\pi^{1/2}|\gamma_{12}^i|^2(n_2 - n_1)/2\omega_0'^3 \equiv \lambda \qquad \lambda > 0 \quad \text{for inversion} \tag{2.22b}$$

$$\tfrac{1}{4}\alpha\bar{p}_c(\mathbf{k}\omega_0')/\omega_0' \equiv p \qquad \text{note} \quad p = \alpha/2Q \tag{2.22c}$$

z and z_0' are, respectively, the displacement of the frequency of interest and the empty cavity resonance frequency from the center of the atomic transition line, as a fraction of the Doppler width. λ is a measure of the population inversion or, equivalently, of the pump power while p describes the dissipation of energy in the cavity.

In terms of these variables (2.21) becomes

$$(2\omega_0'^2/\alpha)\mathbf{d}^r(\mathbf{k}\omega) = -[z - z_0' + ip - i\lambda \exp(-z^2) + 2\lambda\pi^{-1/2}f_1(z)]^{-1} \quad (2.23)$$

The criterion for stability of the system is that $\mathbf{d}^r(\mathbf{k}, t - t')$ does not correspond to a growing wave, so that $\mathbf{d}^r(\mathbf{k}, \omega)$ is analytic in the upper half ω plane. Since $\mathbf{d}^r(\mathbf{k}\omega)^{-1}$ is a well behaved function of ω we need only insist that it not have any zeros for $\text{Im}\,\omega > 0$. Clearly for $\lambda < p$ the stability condition is satisfied. There will be an instability threshold at that value of λ where $\mathbf{d}^{-1}(\mathbf{k}\omega)$ has a zero for ω real, and it is easy to show that this zero moves into the unstable region as λ increases. To find this threshold we look for a real number z_0 such that

$$z_0 - z_0' + 2\lambda\pi^{-1/2}f_1(z_0) = 0; \qquad p - \lambda \exp(-z_0^2) = 0 \quad (2.24)$$

Then

$$z_0 - z_0' + 2p\pi^{-1/2} \int_0^{z_0} \exp(y^2)\, dy = 0 \quad (2.24')$$

where we have used the definition (2.15) of $f_1(z)$. The equation for z_0 can easily be solved graphically, giving a value with the same sign as z_0' and smaller amplitude. When the detuning z_0' is small, the linear approximation to the integral in (2.24′) is satisfactory and leads to the evaluation

$$z_0 \approx z_0'/(1 + 2p\pi^{-1/2}) \quad (2.24'')$$

and better values can easily be obtained. Substitution of this value for z_0 in the second of Eqs. (2.24) gives the threshold value of λ. We note that this threshold value of λ and the frequency z_0 where the inverse propagator vanishes agree with the evaluations of numerous authors (*16*, *17*) of the threshold inversion needed for and the frequency of laser oscillation.

For a better understanding of this threshold let us see how the electromagnetic energy density and noise power output depend on the inversion λ. We show in Appendix I that the power absorbed by the cavity, which we may take as output power, is given by

$$\text{Power} = -\sum_{\mathbf{k}} \int_0^\infty \frac{d\omega}{2\pi}\, i\omega^2 \bar{\mathbf{p}}_c(\mathbf{k}\omega)\mathbf{d}^<(\mathbf{k}\omega) \quad (2.25)$$

Using (1.54c), (1.55), and the definition (1.39) we can write

$$\mathbf{d}^<(\mathbf{k}\omega) = \mathbf{d}^r(\mathbf{k}\omega)\mathbf{p}^<(\mathbf{k}\omega)\mathbf{d}^a(\mathbf{k}\omega)$$
$$= \mathbf{p}^<(\mathbf{k}\omega)\bar{\mathbf{d}}(\mathbf{k}\omega)/\omega\bar{\mathbf{p}}(\mathbf{k}\omega)^{19} \tag{2.26}$$

so that the power output from the mode **k** is

$$P_{\mathbf{k}} = -i\int_0^\infty \frac{d\omega}{2\pi}\,\omega\bar{\mathbf{d}}(\mathbf{k}\omega)\bar{\mathbf{p}}_c(\mathbf{k}\omega)\,\frac{\mathbf{p}^<(\mathbf{k}\omega)}{\bar{\mathbf{p}}(\mathbf{k}\omega)} \tag{2.27}$$

But, for a transverse mode, $\int_0^\infty d\omega\omega\,\bar{d}_{ii}(\mathbf{k}\omega) = \pi$ (see (1.41)) and if the linewidth is narrow relative to the variation in the remainder of the integrand we may write

$$P_{\mathbf{k}} = -\tfrac{1}{2}i\bar{\mathbf{p}}_c(\mathbf{k}\omega_0)\mathbf{p}^<(\mathbf{k}\omega_0)/\bar{\mathbf{p}}(\mathbf{k}\omega_0) \tag{2.28}$$

where ω_0 is the line center.

Using (2.5), (2.22), (2.16), (2.19), and (2.23) and the representation[17] $\mathbf{p}_c^<(\omega) = (e^{\beta\omega} - 1)^{-1}i\omega\bar{\mathbf{p}}_c(\omega)$ (cf. (1.58), (1.59)) this is

$$P_{\mathbf{k}} = \frac{2\omega_0^2 p}{\alpha}\left[\frac{p}{e^{\beta\omega_0} - 1} + \frac{\lambda\exp(-z_0^2)n_2}{n_2 - n_1}\right](p - \lambda e^{-z_0^2})^{-1} \tag{2.29}$$

so the instability manifests itself here as a blowup of the noise output power or, equivalently, the noise energy in the system. We remark parenthetically that, using the relations

$$\omega_0 p/\alpha \equiv \Delta\omega_c \tag{2.30a}$$

$$\omega_0[p - \lambda\exp(-z_0^2)]/\alpha \equiv \Delta\omega \tag{2.30b}$$

where $\Delta\omega_c$ is the cavity line halfwidth while $\Delta\omega$ is the narrowed output halfwidth, we may write

$$P_{\mathbf{k}} = \frac{2\omega_0(\Delta\omega_c)^2}{\Delta\omega}\left[\frac{1}{e^{\beta\omega_0} - 1} + \frac{\lambda\exp(-z_0^2)}{p}\,\frac{n_2}{n_2 - n_1}\right] \tag{2.29$'$}$$

Near threshold, where (2.24) almost holds, this is identical to the Townes formula (*16*) relating power output to amplifier linewidth, when factors of 2π are inserted to go from circular frequencies to cycles per second.

As is clear from this computation of the noise power in the system the instability we found above is a property of our approximation rather than the system itself, and reflects our neglect of nonlinear effects which lead to stability. We turn to a discussion of these effects.

[19] We have restricted ourselves to the case where all quantities are diagonal in the polarization indices.

B. Nonlinear Coherent Field Description

We find it convenient to base our nonlinear treatment on Eq. (1.48) for the electromagnetic field expectation value in the presence of an external source

$$\Box\langle A_i^{\zeta}(\mathbf{x})\rangle = j_i^{\zeta}(\mathbf{x}) + \langle J_{ci}^{\zeta}(\mathbf{x})\rangle + i\zeta\gamma_{\alpha\beta}^{i}G_{\beta\alpha}^{\zeta\zeta}(\mathbf{x}, \mathbf{x}^{\zeta}) \tag{2.31}$$

since equations for correlation functions may be derived from this expression by differentiation.

We first assume that the cavity response is linear so that, from (1.49) and (1.52) we may write

$$\langle J_{ci}^{\zeta}(\mathbf{x})\rangle = \int d\mathbf{x}_1 P_{cij}^{\zeta\zeta_1}(\mathbf{x}\mathbf{x}_1)\langle A_j^{\zeta_1}(\mathbf{x}_1)\rangle \tag{2.32}$$

and we will generally take $\mathbf{P}_c$ to be diagonal in i, j. What remains is an evaluation of $\mathbf{G}_{\beta\alpha}$.

From (1.20) it is easy to expand the mass operator $M_{\alpha\beta}(\mathbf{x}\mathbf{x}')$ in terms of $\mathbf{G}$ and field correlation functions. One could then in principle invert (1.19) or (1.23) to express $\mathbf{G}_{\beta\alpha}$ solely in terms of field correlations. In practice of course this can only be done in an approximate way. We shall find it convenient to use the simplest of these approximations, an expansion of $\mathbf{G}_{\beta\alpha}$ in powers of the electromagnetic potential, which should be accurate for a gas laser under moderate excitation. The procedure we use is the following. We write $\mathbf{M}_{\alpha\beta}$ as $\mathbf{M}_{\alpha\alpha}^{0}\delta_{\alpha\beta} + \mathbf{M}_{\alpha\beta}^{1}$ where $\mathbf{M}_{\alpha\alpha}^{0}$ includes those terms describing natural level width and pumping. Defining

$$\hat{\mathbf{G}}_{\alpha\beta} \equiv (\mathbf{G}_{\alpha}^{0-1} - \mathbf{M}_{\alpha\alpha}^{0})^{-1}\delta_{\alpha\beta} \equiv \hat{\mathbf{G}}_{\alpha}\delta_{\alpha\beta} \tag{2.33a}$$

we write

$$\mathbf{G}_{\alpha\beta} = \hat{\mathbf{G}}_{\alpha}\delta_{\alpha\beta} + \hat{\mathbf{G}}_{\alpha}\mathbf{M}_{\alpha\beta}^{1}\hat{\mathbf{G}}_{\beta} + \hat{\mathbf{G}}_{\alpha}\mathbf{M}_{\alpha\gamma}^{1}\hat{\mathbf{G}}_{\gamma}\mathbf{M}_{\gamma\beta}^{1}\hat{\mathbf{G}}_{\beta} + \cdots \tag{2.33b}$$

Note that while $\mathbf{G}$ includes the effects of saturation, $\hat{\mathbf{G}}$ does not and can be described in terms of the unsaturated level occupations as in (2.2), (2.3), (2.4). The expansion of $\mathbf{G}_{\alpha\beta}$ is carried out in Appendix II to order $(\gamma\mathbf{A})^3$. Combining the result (A.10) with (2.31) and (2.32), we have

$$a_i^{\zeta\zeta_1}(\mathbf{x}\mathbf{x}_1)\langle A_i^{\zeta_1}(\mathbf{x}_1)\rangle + b_{ijkl}^{\zeta\zeta_1\zeta_2\zeta_3}(\mathbf{x}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3)\langle A_j^{\zeta_1}(\mathbf{x}_1)A_k^{\zeta_2}(\mathbf{x}_2)A_l^{\zeta_3}(\mathbf{x}_3)\rangle = j_i^{\zeta}(\mathbf{x}) \tag{2.34a}$$

where

$$a_i^{\zeta\zeta_1}(\mathbf{x}\mathbf{x}_1) \equiv \Box\delta^{\zeta\zeta_1}(\mathbf{x} - \mathbf{x}_1) - \mathrm{P}_{cii}^{\zeta\zeta_1}(\mathbf{x}\mathbf{x}_1) + i\zeta\,|\,\gamma_{\alpha\beta}^{i}\,|^2\,\hat{G}_{\beta}^{\zeta\zeta_1}(\mathbf{x}\mathbf{x}_1)\hat{G}_{\alpha}^{\zeta_1\zeta}(\mathbf{x}_1\mathbf{x}) \tag{2.34b}$$

$$b_{ijkl}^{\zeta\zeta_1\zeta_2\zeta_3}(\mathbf{x}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3) \equiv i\zeta\gamma_{\alpha\beta}^{i}\gamma_{\beta\sigma}^{j}\gamma_{\sigma\lambda}^{k}\gamma_{\lambda\alpha}^{l}\hat{G}_{\beta}^{\zeta\zeta_1}(\mathbf{x}\mathbf{x}_1) \times \hat{G}_{\sigma}^{\zeta_1\zeta_2}(\mathbf{x}_1\mathbf{x}_2)\hat{G}_{\lambda}^{\zeta_2\zeta_3}(\mathbf{x}_2\mathbf{x}_3)\hat{G}_{\alpha}^{\zeta_3\zeta}(\mathbf{x}_3\mathbf{x}) \tag{2.34c}$$

The quantity $a_i^{\zeta\zeta_1}(\mathbf{xx}_1)$ has already been computed as the inverse of $\mathbf{D}$ in our linear analysis. The quantity $\mathbf{b}$ describes the nonlinear nonlocal coupling of the field to itself. We shall henceforth consider (2.34) as the basic equation for our discussion. All our further efforts will be to the end of analyzing the system described by this equation.

The laser has been treated elsewhere (*5*, *18*, *19*) by considering the above threshold electromagnetic field as a classical or quantum mechanical oscillation with a well defined frequency and phase. We will first analyze (2.34) under the assumption that this is the correct description. We ask for the conditions under which a stable electromagnetic mode can exist in the cavity. Then we will consider whether, in the presence of this mode, the correlation function $\mathbf{D}^{\zeta\zeta'}$ satisfies the criterion for stability of the whole system against small disturbances.

By the existence of a stable mode of the system we mean a nonvanishing field expectation value $\langle \mathbf{A}(\mathbf{r}t)\rangle$ when the external source $\mathbf{j}^{\pm}(\mathbf{r}t)$ is set to zero. We will further assume that the coherent part of the field dominates the incoherent part so that we may write the three-field expectation value in (2.34a) as the product of single field expectations. Then we have

$$a^{\zeta\zeta_1}(\mathbf{xx}_1)\langle A_i^{\zeta_1}(\mathbf{x}_1)\rangle + b_{ijkl}^{\zeta\zeta_1\zeta_2\zeta_3}(\mathbf{xx}_1\mathbf{x}_2\mathbf{x}_3)\langle A_j^{\zeta_1}(\mathbf{x}_1)\rangle\langle A_k^{\zeta_2}(\mathbf{x}_2)\rangle\langle A_l^{\zeta_3}(\mathbf{x}_3)\rangle = 0 \quad (2.35)$$

Now from (1.5) when $j^{\pm} \to 0$, $\langle A_i^{+}(\mathbf{x})\rangle = \langle A_i^{-}(\mathbf{x})\rangle = \langle A_i(\mathbf{x})\rangle$. Then the relevant coefficients in (2.35) are

$$a^{r}(\mathbf{xx}_1) \equiv \sum_{\zeta_1} a^{\zeta\zeta_1}(\mathbf{xx}_1) \quad (2.36\text{a})$$

$$b_{ijkl}^{r}(\mathbf{xx}_1\mathbf{x}_2\mathbf{x}_3) \equiv \sum_{\zeta_1\zeta_2\zeta_3} b_{ijkl}^{\zeta\zeta_1\zeta_2\zeta_3}(\mathbf{xx}_1\mathbf{x}_2\mathbf{x}_3) \quad (2.36\text{b})$$

where, as indicated by our notation, no ζ dependence remains. The superscript r on $\sum_{\zeta_1} a^{\zeta\zeta_1}$ reflects the fact that $a^{\zeta\zeta_1}$ has the matrix form (1.53) and that a^r is the inverse of the retarded propagator (2.17) in the linear approximation. The notation $\mathbf{b}^r$ has a similar origin. To evaluate $\mathbf{b}^r$ we first note the relation which follows from (1.10) and (1.11d)

$$G_\alpha^{\zeta\zeta_1}G_\beta^{\zeta_1\zeta'} = g_\alpha^{\ r}g_\beta^{\ r}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + g_\alpha^{\ r}\begin{pmatrix}g_\beta^{\ <} & -g_\beta^{\ <}\\ g_\beta^{\ >} & -g_\beta^{\ >}\end{pmatrix} + \begin{pmatrix}g_\alpha^{\ <} & -g_\alpha^{\ <}\\ g_\alpha^{\ >} & -g_\alpha^{\ >}\end{pmatrix}g_\beta^{\ a} \quad (2.37)$$

With (2.34c) this yields

$$\begin{aligned}b_{ijkl}^{r}(\mathbf{xx}_1\mathbf{x}_2\mathbf{x}_3) = {}& i\gamma_{\alpha\beta}^{i}\gamma_{\beta\sigma}^{j}\gamma_{\sigma\lambda}^{k}\gamma_{\lambda\alpha}^{l}[g_\beta^{\ r}(\mathbf{xx}_1)g_\sigma^{\ r}(\mathbf{x}_1\mathbf{x}_2)g_\lambda^{\ r}(\mathbf{x}_2\mathbf{x}_3)g_\alpha^{\ <}(\mathbf{x}_3\mathbf{x})\\ &+ g_\beta^{\ r}(\mathbf{xx}_1)g_\sigma^{\ r}(\mathbf{x}_1\mathbf{x}_2)g_\lambda^{\ <}(\mathbf{x}_2\mathbf{x}_3)g_\alpha^{\ a}(\mathbf{x}_3\mathbf{x}) + g_\beta^{\ r}(\mathbf{xx}_1)g_\sigma^{\ <}(\mathbf{x}_1\mathbf{x}_2)g_\lambda^{\ a}(\mathbf{x}_2\mathbf{x}_3)g_\alpha^{\ a}(\mathbf{x}_3\mathbf{x}) \quad (2.38)\\ &+ g_\beta^{\ <}(\mathbf{xx}_1)g_\sigma^{\ a}(\mathbf{x}_1\mathbf{x}_2)g_\lambda^{\ a}(\mathbf{x}_2\mathbf{x}_3)g_\alpha^{\ a}(\mathbf{x}_3\mathbf{x})]\end{aligned}$$

Using the approximate forms (2.2), (2.3), (2.9) for $g^<$ and g^r this can be written in a more explicit fashion. In Appendix III we perform the straightforward but tedious reduction of (2.38). In combination with our previous evaluation of a^r, (2.8 ff), (2.17), this enables us to write (2.35) as

$$
\begin{aligned}
0 = {} & [\omega^2 - c^2k^2 + p_c^r(\mathbf{k}\omega)]\langle A_i(\mathbf{k}\omega)\rangle \\
& + |\gamma^i_{\alpha\beta}|^2 \frac{\alpha\pi^{1/2}}{c|\mathbf{k}|} \int \frac{d^3\mathbf{K}}{(2\pi)^3} (n_\alpha(\mathbf{K}) - n_\beta(\mathbf{K}))^{20} \Bigg\{ f\left[\frac{\alpha(\omega - \omega_{\beta\alpha})}{c|\mathbf{k}|}\right] \langle A_i(\mathbf{k} - \mathbf{K},\omega)\rangle \\
& + 2i\,|\gamma^i_{\alpha\beta}|^2 \exp[-\alpha^2(\omega - \omega_{\beta\alpha})^2/c^2k^2] \int (2\pi)^{-12}\, d\omega'\, d\omega''\, d\omega'''\, d\mathbf{k}'\, d\mathbf{k}''\, d\mathbf{k}''' \\
& \times (2\pi)^4 \delta(\omega - \omega' - \omega'' - \omega''')\delta((\mathbf{k} - \mathbf{k}' - \mathbf{k}'' - \mathbf{k}''' - \mathbf{K}) \\
& \qquad \cdot [(\omega - \omega' + 2i\gamma_\alpha)^{-1} + (\omega - \omega' + 2i\gamma_\beta)^{-1}] \\
& \times \Bigg[\frac{\langle A_i(\mathbf{k}'\omega')\rangle^+ \langle A_i(\mathbf{k}''\omega'')\rangle^+ \langle A_i(\mathbf{k}'''\omega''')\rangle^-}{\omega + \omega''' - 2\omega_{\beta\alpha} + 2i\gamma_{\alpha\beta}} \\
& \qquad + \frac{\langle A_i(\mathbf{k}'\omega')\rangle^+ \langle A_i(\mathbf{k}''\omega'')\rangle^- \langle A_i(\mathbf{k}'''\omega''')\rangle^+}{\omega + \omega'' + 2i\gamma_{\alpha\beta}} \Bigg] \Bigg\}
\end{aligned} \tag{2.39}
$$

where we have specialized to a single direction of polarization. The $(+)$ or $(-)$ sign appended to $\langle A_i(\mathbf{k}'\omega')\rangle$ here specifies that $\mathbf{k}'$ is very close to $+\mathbf{k}$ or $-\mathbf{k}$ respectively.

Equation (2.39) is the same as that derived by Lamb[21] except for the exponential factor in the cubic term (which might as well be set to unity in our approximation, see note after (A.14)) and the fact that Lamb insists on standing wave solutions so that $\langle \mathbf{A}(\mathbf{k}'\omega')\rangle = e^{i\phi(\mathbf{k}')}\langle \mathbf{A}(-\mathbf{k}', \omega')\rangle$ for all $\mathbf{k}'$.

Using this equation Lamb finds conditions for the existence of one or several modes and the interactions among modes. It would be pointless to echo his arguments here. We will only look at the condition for the existence of a single mode as that will be our main concern in what follows.

When $\langle \mathbf{A}(\mathbf{k}\omega)\rangle$ is nonzero for only one mode $(\pm\mathbf{k}_0, \pm\omega_0)$ we may set $\omega' = -\omega'' = \omega''' = \omega$ in (2.39). Furthermore, only $\mathbf{K} = 0$ remains. Then we may factor out one power of $\langle \mathbf{A}(\mathbf{k}\omega)\rangle$, use the definition (2.20) of the cavity parameters, and return to the dimensionless variables defined in (2.22) to find the criterion for the nonvanishing of $\langle \mathbf{A}(\mathbf{k}_0\omega_0)\rangle$

$$
\begin{aligned}
& z_0 - z_0' + 2\lambda\pi^{-1/2} f_1(z_0) + i(p - \lambda \exp(-z_0^2)) \\
& \qquad - \lambda \exp(-z_0^2)[|a^-|\Gamma_{12}/(z_0 + i\Gamma_{12}) - i|a^+|] = 0
\end{aligned} \tag{2.40}
$$

[20] From (2.1) and (2.2) we recall that the K dependence refers to spatial inhomogeneities in the pumped system.

[21] See ref. 5, especially Eqs. (50) and (76).

where we have defined

$$\Gamma_{12} \equiv \alpha\gamma_{12}/\omega_0' \tag{2.41}$$

$$| a^{\pm} | \equiv | \gamma_{12}^{i} |^2 | \langle A_i(\pm\mathbf{k}_0, \omega_0)\rangle |^2/2\gamma_1\gamma_2 \tag{2.42}$$

Γ_{12} is the ratio of the natural linewidth to the Doppler width and we recall that z_0 and z_0' are the displacements of the oscillation and empty cavity resonance frequencies from the atomic line center as fractions of the Doppler width. The quantities p and λ represent cavity absorption and unsaturated level inversion respectively while $| a^{\pm} |$ is a normalized field intensity. This particular expression can be shown to be the expansion parameter in our series development (2.33) of $\mathbf{G}_{\alpha\beta}$ so we expect that our conclusions will be valid as long as the calculated value of $| a^{\pm} |$ is small compared to unity.

If our system is invariant under $\mathbf{k} \to -\mathbf{k}$, or if we insist on standing wave boundary conditions, we have $| a^+ | = | a^- | \equiv | a |$. Then the oscillation criterion is that found by Lamb. If the system is so arranged that propagation is possible for only one of the traveling waves which make up the standing wave we must set $| a^- |$ to zero. In this case (2.40) becomes

$$\begin{aligned} p &= \lambda \exp(-z_0^2)(1 - | a |) \\ z_0 - z_0' + 2\lambda\pi^{-1/2}f_1(z_0) &= 0 \end{aligned} \tag{2.43}$$

We note that there is a threshold at the same place the linear approximation (2.21) became unsatisfactory and that above this threshold the intensity $| a |$ grows with increasing λ. From the relation

$$z_0 - z_0' = -2p\pi^{-1/2}(1 - | a |)^{-1} \int_0^{z_0} \exp(y^2)\, dy \tag{2.43$'$}$$

we see that as λ and thus $| a |$ increases the resonant frequency z_0 moves closer to zero so there is power dependent frequency pulling in this case.

III. FLUCTUATIONS ACCOMPANYING COHERENT FIELD

Now that we have found the frequency and amplitude of the pure oscillation we assume to be present when our linear theory breaks down we turn to a calculation of the correlation function $D^{\zeta\zeta'}(\mathbf{xx}')$ in the presence of this oscillation. From (2.34) and the assumption we made to get to (2.35) that $\langle \mathbf{AAA} \rangle$ is dominated by $\langle \mathbf{A} \rangle \langle \mathbf{A} \rangle \langle \mathbf{A} \rangle$ we find

$$\begin{aligned} \{a^{\zeta\zeta_1}(\mathbf{xx}_1) + \sum_{\zeta_2\zeta_3} [b^{\zeta\zeta_1\zeta_2\zeta_3}(\mathbf{xx}_1\mathbf{x}_2\mathbf{x}_3) + b^{\zeta\zeta_2\zeta_1\zeta_3}(\mathbf{xx}_2\mathbf{x}_1\mathbf{x}_3) + b^{\zeta\zeta_2\zeta_3\zeta_1}(\mathbf{xx}_2\mathbf{x}_3\mathbf{x}_1)] \\ \cdot\langle A(\mathbf{x}_2)\rangle\langle A(\mathbf{x}_3)\rangle\} D^{\zeta_1\zeta'}(\mathbf{x}_1\mathbf{x}') = \delta^{\zeta\zeta'}(\mathbf{x} - \mathbf{x}') \end{aligned} \tag{3.1}$$

Equation (3.1) is in the general form (1.51) so that the coefficient of $D^{\zeta_1\zeta'}$ must be expressible in the form (1.53), while corresponding to (1.54) we have

$$\{a^r(\mathbf{x}\mathbf{x}_1) + [b^r(\mathbf{x}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3) + b^r(\mathbf{x}\mathbf{x}_2\mathbf{x}_1\mathbf{x}_3) + b^r(\mathbf{x}\mathbf{x}_2\mathbf{x}_3\mathbf{x}_1)]\langle A(\mathbf{x}_2)\rangle\langle A(\mathbf{x}_3)\rangle\} \times d^r(\mathbf{x}_1\mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \tag{3.2a}$$

$$d^<(\mathbf{x}\mathbf{x}') = -d^r(\mathbf{x}\mathbf{x}_1)[a^<(\mathbf{x}_1\mathbf{x}_2) + B^<(\mathbf{x}_1\mathbf{x}_2)]\, d^a(\mathbf{x}_2\mathbf{x}') \tag{3.2b}$$

where a^r and b^r are defined in (2.36) and

$$a^<(\mathbf{x}\mathbf{x}_1) \equiv -a^{+-}(\mathbf{x}\mathbf{x}_1) \tag{3.3a}$$

$$B^<(\mathbf{x}\mathbf{x}_1) \equiv -\sum_{\zeta_2\zeta_3} [b^{+-\zeta_2\zeta_3}(\mathbf{x}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3) + b^{+\zeta_2-\zeta_3}(\mathbf{x}\mathbf{x}_2\mathbf{x}_1\mathbf{x}_3) + b^{+\zeta_2\zeta_3-}(\mathbf{x}\mathbf{x}_2\mathbf{x}_3\mathbf{x}_1)] \cdot\langle A(\mathbf{x}_2)\rangle\langle A(\mathbf{x}_3)\rangle \tag{3.3b}$$

We further define

$$B^r(\mathbf{x}\mathbf{x}_1) \equiv [b^r(\mathbf{x}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3) + b^r(\mathbf{x}\mathbf{x}_2\mathbf{x}_1\mathbf{x}_3) + b^r(\mathbf{x}\mathbf{x}_2\mathbf{x}_3\mathbf{x}_1)][A(\mathbf{x}_2)\rangle\langle A(\mathbf{x}_3)\rangle \tag{3.3c}$$

In order to proceed further with Eq. (3.2) we introduce Fourier transforms

$$\mathbf{D}(\mathbf{x}\mathbf{x}') \equiv \int \frac{d^3\mathbf{k}\, d^3\mathbf{k}'\, d\omega\, d\omega'}{(2\pi)^8} \mathbf{D}(\mathbf{k}\omega, \mathbf{k}'\omega')e^{i[(\mathbf{k}\cdot\mathbf{r}-\mathbf{k}'\cdot\mathbf{r}')-(\omega t-\omega' t')]} \tag{3.4}$$

with similar expressions for $\mathbf{a}(\mathbf{x}\mathbf{x}')$ and $\mathbf{B}(\mathbf{x}\mathbf{x}')$. Then we may rewrite (3.2a) as

$$(2\pi)^{-4}\int d\mathbf{k}_1\, d\omega_1[a^r(\mathbf{k}\omega, \mathbf{k}_1\,\omega_1) + B^r(\mathbf{k}\omega, \mathbf{k}_1\,\omega_1)]\, d^r(\mathbf{k}_1\,\omega_1\,, \mathbf{k}'\omega') = (2\pi)^4\delta(\omega - \omega')\delta(\mathbf{k} - \mathbf{k}') \tag{3.5}$$

Now from its construction, (2.34b) in terms of quantities possessing time and space translational invariance $\mathbf{a}(\mathbf{k}\omega, \mathbf{k}_1\omega_1)$ satisfies[22]

$$\mathbf{a}(\mathbf{k}\omega, \mathbf{k}_1\omega_1) = (2\pi)^4\mathbf{a}(\mathbf{k}\omega)\delta(\mathbf{k} - \mathbf{k}_1)\delta(\omega - \omega_1) \tag{3.6}$$

This is not, however, the case for $\mathbf{B}(\mathbf{k}\omega, \mathbf{k}_1\omega_1)$. If we write, suppressing spatial variables for the moment,

$$\mathbf{b}(tt_1\, t_2\, t_3) \equiv (2\pi)^{-4}\int d\omega\, d\omega_1\, d\omega_2\, d\omega_3\, \mathbf{b}(\omega\omega_1\,\omega_2\,\omega_3)e^{-i[\omega t-\omega_1 t_1-\omega_2 t_2-\omega_3 t_3]} \tag{3.7a}$$

the construction (2.34c) of $\mathbf{b}$ insures that

$$\mathbf{b}(\omega\omega_1\omega_2\omega_3) \propto \delta(\omega - \omega_1 - \omega_2 - \omega_3) \tag{3.7b}$$

Then, from (3.3), $\mathbf{B}(\omega, \omega_1)$ has contributions for $\omega_1 = \omega \pm \omega' \pm \omega''$ where ω' and ω'' are frequencies of the oscillating field present in the system. Similarly,

[22] For the remainder of the discussion we ignore spatial inhomogeneities.

replacing the spatial dependence, $\mathbf{B}(\mathbf{k}\omega, \mathbf{k}_1\omega_1)$ exists when $\mathbf{k}_1 = \mathbf{k} \pm \mathbf{k}' \pm \mathbf{k}''$ where $\mathbf{k}'$ is the wavenumber associated with ω' and $\mathbf{k}''$ with ω''.

Returning to (3.5) we see that this behavior of $\mathbf{B}(\mathbf{k}\omega, \mathbf{k}_1\omega_1)$ greatly complicates the equation for $\mathcal{C}$. Instead of a single equation for $\mathcal{C}(\mathbf{k}\omega, \mathbf{k}_1\omega_1)$, this quantity is now coupled to quantities of the form $\mathcal{C}(\mathbf{k} \pm \mathbf{k}' \pm \mathbf{k}'', \omega \pm \omega' \pm \omega''; \mathbf{k}_1\omega_1)$ with all possible combinations of $\mathbf{k}'$, $\mathbf{k}''$, ω', and ω''.

We remark that this complication is a natural consequence of the presence of steady modes in the system. In a crystal the breakdown of spatial translational invariance by imposition of a periodic structure leads to "umklapp" processes where an electron may freely change its momentum by a reciprocal lattice vector. In our system we have a periodic structure in time as well as in space, so that jumps in momentum and energy of $\mathbf{k}'$, ω' are allowed. Because of symmetry requirements (see (A.9), (A.10)) only double jumps occur, leading to the coupled equations discussed above.

We simplify the above equations somewhat by restricting ourselves to a single mode in oscillation, with wave number $\mathbf{k}_0$ and frequency ω_0. Then $\mathbf{B}(\mathbf{k}\omega, \mathbf{k}_1\omega_1)$ has terms with $\omega_1 = \omega, \omega + 2\omega_0, \omega - 2\omega_0 \mathbf{k}_1 = \mathbf{k}, \mathbf{k} + 2\mathbf{k}_0, \mathbf{k} - 2\mathbf{k}_0$. A further simplification results if we notice that $\mathbf{b}(\omega\omega_1\omega_2\omega_3)$ is only appreciable when each frequency is close to an atomic resonance. If we ignore the small off-resonance terms we can write

$$\begin{aligned}\mathbf{B}(\mathbf{k}\omega, \mathbf{k}_1\omega_1) &= (2\pi)^4[\mathbf{B}_1(\mathbf{k}\omega)\delta(\mathbf{k}_1 - \mathbf{k})\delta(\omega_1 - \omega) \\ &+ \mathbf{B}_2(\mathbf{k}\omega)\delta(\mathbf{k}_1 - \mathbf{k})\delta(\omega_1 - \omega + 2\omega_0) + \mathbf{B}_3(\mathbf{k}\omega)\delta(\mathbf{k}_1 - \mathbf{k} + 2\mathbf{k}_0)\delta(\omega_1 - \omega) \\ &+ \mathbf{B}_4(\mathbf{k}\omega)\delta(\mathbf{k}_1 - \mathbf{k} + 2\mathbf{k}_0)\delta(\omega_1 - \omega + 2\omega_0)]\end{aligned} \tag{3.8}$$

where ω_0 and $\mathbf{k}_0$ have the signs of ω and $\mathbf{k}$. Note that this dropping of off-resonance terms is consistent with the approximation we made in going from (2.39) to (2.40), of setting $\omega' = -\omega'' = \omega''' = \omega$. We go one step further by asserting that we have travelling waves in one direction only. Then a unique $\mathbf{k}_0$ is associated with ω_0 and (3.8) is replaced by

$$\begin{aligned}\mathbf{B}(\mathbf{k}\omega, \mathbf{k}_1\omega_1) &= (2\pi)^4[\mathbf{B}(\mathbf{k}\omega)\delta(\mathbf{k}_1 - \mathbf{k})\delta(\omega_1 - \omega) \\ &+ \bar{\mathbf{B}}(\mathbf{k}\omega)\delta(\mathbf{k}_1 - \mathbf{k} + 2\mathbf{k}_0)\delta(\omega_1 - \omega + 2\omega_0)]\end{aligned} \tag{3.9}$$

If we restrict ourselves to the radiation in the resonant mode, $\mathbf{k}, \mathbf{k}_1 = \pm\mathbf{k}_0$, and the choice of $(\pm)$ is dictated by the particular value of ω. Then we will henceforth disregard the $\mathbf{k}$ variable as redundant. Finally we remark that (see (A.15)) $\mathbf{b}(\omega\omega_1\omega_2\omega_3)$ is small unless $\omega_1 \approx -\omega_2 \approx \omega_3 \approx \omega$ so that, from (3.3) and (3.9),

$$\begin{aligned}B^{\zeta\zeta_1}(\omega) &= \sum_{\zeta_2\zeta_3} [b^{\zeta\zeta_1\zeta_2\zeta_3}(\omega, \omega, -\omega_0, \omega_0) - b^{\zeta\zeta_2\zeta_3\zeta_1}(\omega, \omega_0, -\omega_0, \omega)] \\ &\cdot |\langle A(\omega_0)\rangle|^2 \equiv 2b^{\zeta\zeta_1}(\omega)|\langle A(\omega_0)\rangle|^2\end{aligned} \tag{3.10a}$$

$$\bar{B}^{\zeta\zeta_1}(\omega) = \sum_{\zeta_2\zeta_3} b^{\zeta\zeta_2\zeta_1\zeta_3}(\omega,\omega_0\,,\,\omega - 2\omega_0\,,\,\omega_0)\,\langle A(\omega_0)\rangle^2 \equiv \bar{b}^{\zeta\zeta_1}(\omega)\langle A(\omega_0)\rangle^2 \quad (3.10b)$$

Taking account of (3.3)–(3.10) we may now write Eq. (3.2a) in the matrix form

$$\begin{pmatrix} a^r(\omega) + 2b^r(\omega)\,|\,\langle A\rangle\,|^2 & \bar{b}^r(\omega)\langle A(\omega_0)\rangle^2 \\ \bar{b}^r(\omega - 2\omega_0)\langle A(-\omega_0)\rangle^2 & a^r(\omega - 2\omega_0) + 2b^r(\omega - 2\omega_0)\,|\,\langle A\rangle\,|^2 \end{pmatrix} (\mathbf{D}^r(\omega)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.11a)$$

where

$$(\mathbf{D}^{\zeta\zeta'}(\omega)) \equiv \begin{pmatrix} D^{\zeta\zeta'}(\omega) & \bar{D}^{\zeta\zeta'}(\omega) \\ \bar{D}^{\zeta\zeta'}(\omega - 2\omega_0) & D^{\zeta\zeta'}(\omega - 2\omega_0) \end{pmatrix} \quad (3.11b)$$

and

$$D^{\zeta\zeta'}(\omega,\,\omega') = 2\pi[D^{\zeta\zeta'}(\omega)\delta(\omega - \omega') + \bar{D}^{\zeta\zeta'}(\omega)\delta(\omega' - \omega + 2\omega_0)] \quad (3.11c)$$

Defining $\omega - 2\omega_0 \equiv \omega_-$, $\langle A(\omega_0)\rangle \equiv \langle A_+\rangle$, $\langle A(-\omega_0\rangle \equiv \langle A_-\rangle$, (3.2b) becomes

$$(\mathbf{D}^<(\omega)) = -(\mathbf{D}^r(\omega)) \begin{pmatrix} a^<(\omega) + 2b^<(\omega)\,|\,\langle A\rangle\,|^2 & \bar{b}^<(\omega)\langle A_+\rangle^2 \\ \bar{b}^<(\omega_-)\langle A_-\rangle^2 & a^<(\omega_-) + 2b^<(\omega_-)\,|\,\langle A\rangle\,|^2 \end{pmatrix} \cdot(\mathbf{D}^a(\omega)) \quad (3.11d)$$

The quantities needed for the solution of (3.11a) have already been evaluated. For convenience we summarize them here. From (2.34), (2.36), (2.6), (2.17), and (2.23) we have

$$\begin{aligned} a^r(\omega) &= -(2\omega_0'^2/\alpha) \\ &\quad\cdot\{z - z_0' + 2\lambda\pi^{-1/2}f_1(z) + i(p - \lambda\exp(-z^2))\} \\ a^r(\omega - 2\omega_0) &= (2\omega_0'^2/\alpha)\{z + z_0' - 2z_0 + 2\lambda\pi^{-1/2}f_1(2z_0 - z) \\ &\quad + i(p - \lambda\exp[-(2z_0 - z)^2])\} \end{aligned} \quad (3.12a)$$

where ω is taken to be positive and close to ω_0 .

From (A.15) and (2.22) we can find, with $\delta \equiv z - z_0$

$$\begin{aligned} b^r(\omega) &= (2\omega_0'^2/\alpha)\lambda'\exp(-z^2)[\Gamma_1/(\delta + 2i\Gamma_1) + \Gamma_2/(\delta + 2i\Gamma_2)] \\ b^r(\omega - 2\omega_0) &= -(2\omega_0'^2/\alpha)\lambda'\exp[-(2z_0 - z)^2] \\ &\quad\cdot[\Gamma_1/(\delta + 2i\Gamma_1) + \Gamma_2/(\delta + 2i\Gamma_2)] \end{aligned} \quad (3.12b)$$

$$\begin{aligned} \bar{b}^r(\omega) &= (2\omega_0'^2/\alpha)i\lambda'\exp(-z^2)4\Gamma_1\Gamma_2(\delta + 2i\Gamma_1)^{-1}(\delta + 2i\Gamma_2)^{-1} \\ \bar{b}^r(\omega - 2\omega_0) &= -(2\omega_0'^2/\alpha)i\lambda'\exp[-(2z_0 - z)^2] \\ &\quad\cdot 4\Gamma_1\Gamma_2(\delta + 2i\Gamma_1)^{-1}(\delta + 2i\Gamma_2)^{-1} \end{aligned} \quad (3.12c)$$

where, in analogy to (2.41), $\Gamma_1 \equiv \alpha\gamma_1/\omega_0'$ and

$$\lambda' \equiv \lambda |\gamma_{12}^i|^2/2\gamma_1\gamma_2 \tag{3.13}$$

so that

$$\lambda' |\langle A(\omega_0)\rangle|^2 = \lambda |a|$$

(see (2.42)). Though we might attack (3.11a) directly with the use of (3.12) it is sufficient for our purposes to consider values of ω very close to ω_0 and expand (3.11a) in the difference. Now from the resonance condition (2.43)

$$a^r(\omega) + b^r(\omega)|\langle A(\omega_0)\rangle|^2 \approx -i\omega_0\xi - 2\omega_0(\omega - \omega_0) \tag{3.14}$$

Although ξ vanishes we have inserted it in anticipation of later arguments. Then to lowest order in $(\omega - \omega_0)$ we find

$$(\mathbf{D}^r(\omega)) = -\frac{1}{4\omega_0 |\langle A\rangle|^2(\omega - \omega_0 + \frac{1}{2}i\xi)} \begin{pmatrix} |\langle A\rangle|^2 & \langle A_+\rangle^2 \\ -\langle A_-\rangle^2 & -|\langle A\rangle|^2 \end{pmatrix} \tag{3.15}$$

so each of the propagators $d^r(\omega)$, $\bar{d}^r(\omega)$ has a pole at $\omega = \omega_0$.

This is, at first glance, a very satisfactory situation. It seems to describe a system where, as the inversion is increased, a threshold occurs for the appearance of a coherent field, signalling its appearance by an apparent instability, (2.23) (2.24), at a certain frequency. The coherent field grows with increasing inversion, (2.40), (2.43), in just such a way that the system always remains on the edge of stability, (3.15). This type of behavior has been found to apply rigorously in a model of a ferromagnetic phase transition discussed by Mermin (*20*). It is also the basis for a description of superconductors (*21*) and superfluids (*22*) where the analogue of increasing population inversion is the lowering of the temperature below its transition value. In the present case, however, it is just this pole in the propagator which will force us to conclude that the foregoing analysis is unsatisfactory. To see this we go on to a discussion of (3.11d), which determines the incoherent field intensity.

Using the relation

$$d^a(\omega, \omega') = d^r(\omega', \omega)^* \tag{3.16}$$

which follows from the definitions (1.7) and (3.4), and in view of (3.15), (3.11d) may be written

$$(\mathbf{D}^<(\omega)) = -\frac{1}{16\omega_0^2[(\omega - \omega_0)^2 + \frac{1}{4}\xi^2]} \frac{-i\phi^<(\omega_0)}{|\langle A\rangle|^2} \begin{pmatrix} |\langle A\rangle|^2 & -\langle A_+\rangle^2 \\ -\langle A_-\rangle^2 & |\langle A\rangle|^2 \end{pmatrix} \tag{3.17}$$

where

$$\begin{aligned} -i\phi^<(\omega_0) \equiv \{&a^<(\omega_0) + a^<(-\omega_0) \\ &+ |\langle A\rangle|^2[2b^<(\omega_0) + 2b^<(-\omega_0) + \bar{b}^<(\omega_0) + \bar{b}^<(-\omega_0)]\} \end{aligned} \tag{3.18}$$

Now the time average of the energy contained in electromagnetic field fluctuations in the mode of interest is given by (see (1.1), (1.7))

$$\langle \mathcal{E} \rangle = -2(2\pi)^{-1} \int_0^\infty d\omega \; i\omega^2 \, d^<(\omega) \tag{3.19}$$

which is easily evaluated from (3.17) and (3.11) to be

$$\langle \mathcal{E} \rangle = \phi^<(\omega_0)/8\xi \tag{3.19$'$}$$

But the condition for the existence of a stable mode is that $\xi = 0$ (cf. (3.14)). Then, unless $\phi^<(\omega_0)$ vanishes, the energy content of this mode is infinite! We will shortly verify that $\phi^<(\omega_0)$ is not zero.

There are two possibilities which suggest themselves at this point for removing this unacceptable infinite energy. First, the starting point of our calculation was the replacement of $\langle AAA \rangle$ by $\langle A \rangle \langle A \rangle \langle A \rangle$ (see (2.34), (2.35)). This entails the neglect of just those terms which we have computed to diverge. It might be expected that the inclusion of these fluctuations and subsequent recalculation of them in a self-consistent manner would yield a finite result for the energy density. A related possibility is that we have made too many approximations in the steps leading to (3.19). A more accurate calculation might replace ξ in (3.17) and thus (3.19) by some finite number. In other words, the electromagnetic field propagator would show a finite lifetime even though a stable mode of the field were able to persist. It can be shown quite generally, however, that neither of these possibilities is the case and that this infinite energy density is an inescapable consequence of the presence of a truly resonant mode. The proof follows from the time-translation invariance of the system as a whole and is related to Goldstone's theorem.[23]

Because of the stationarity of the laser environment there is complete invariance under a redefinition of the time variable $t \to t' \equiv t + \alpha$. In the equation of motion (1.48) for $\langle \mathbf{A}(\mathbf{r}t) \rangle$ in the presence of an external current $\mathbf{j}(\mathbf{r}t)$ this invariance means that replacing $\mathbf{j}(\mathbf{r}t)$ by $\mathbf{j}'(\mathbf{r}t) \equiv \mathbf{j}(\mathbf{r}, t + \alpha)$ changes the solution $\langle \mathbf{A}(\mathbf{r}t) \rangle$ into $\langle \mathbf{A}(\mathbf{r}t) \rangle' = \langle \mathbf{A}(\mathbf{r}, t + \alpha) \rangle$. In terms of Fourier transforms this is

$$\delta \mathbf{j}(\omega') = \mathbf{j}(\omega')(e^{-i\omega'\alpha} - 1) \quad \text{leads to} \quad \delta\langle \mathbf{A}(\omega) \rangle = \langle \mathbf{A}(\omega) \rangle (e^{-i\omega\alpha} - 1).$$

But from (3.4) and (1.8)

$$\lim_{j \to 0} \delta\langle \mathbf{A}(\mathbf{k}, \omega) \rangle / \delta \mathbf{j}(\mathbf{k}'\omega') = \mathbf{D}(\mathbf{k}\omega, \mathbf{k}'\omega') \tag{3.20}$$

[23] See ref. *23*. A proof similar to ours has been used in ref. *22* to show that the superfluid excitation spectrum cannot have a gap. For a number of reasons the difficulties we find do not occur in the homogeneous superfluid. (Actually, related problems do occur in superconductors of the second kind which exhibit resistance, and, more generally, when lifetime effects of the superfluid need to be taken into account.)

so that, for small α,

$$\mathbf{D}^{\zeta\zeta'}(\omega\omega') = \lim_{\mathbf{j}\to 0} \frac{\omega\langle \mathbf{A}^{\zeta}(\omega)\rangle}{\omega' j^{\zeta'}(\omega')} \tag{3.21}$$

But the condition for a resonance is that, for some ω_0, $\langle \mathbf{A}(\omega_0)\rangle$ is nonvanishing when $\mathbf{j}(\omega') = 0$ for all ω'. Then $\mathbf{D}^{\zeta\zeta'}(\omega_0, \omega')$ is infinite for any ω_0 at which resonance holds and for all ω'. In particular $\mathbf{d}^r(\omega, \omega \pm n\omega_0)$ and $\mathbf{d}^a(\omega, \omega \pm n\omega_0)$ will each have a pole at ω_0 so that any calculation of $-\pi^{-1}\int_0^\infty d\omega i\omega^2\, d^<(\omega, \omega)$ from the analogue of (3.11d) will diverge if the corresponding $\phi^<(\omega_0)$ is not zero.

It is easy to see physically just where the difficulty comes from. First let us examine what is implied by the existence of a mode.

The nonvanishing of $\langle A(t)\rangle$ in an ensemble of states set up in the far past means that there is a field excitation which maintains itself coherently through time. For this to occur the *coherent* processes which add to and subtract from the excitation must balance. Incoherent processes do not enter this balance as, due to their random nature, the field they produce has random phase and thus expectation value zero. Then the maintainance of $\langle A\rangle$ requires a balance of the absorption of this field and the emission into the mode *stimulated by the field itself.* Similar remarks apply to the propagator $d^r(\mathbf{k}\omega)$. As we demonstrated in the discussion of g^r ((1.42) and following remarks) and mentioned in the discussion following (1.58) for d^r, the rate of decay of the small excitation described by d^r is given by the difference between absorption and stimulated emission. The meaning of (3.21) is that when absorption balances stimulated emission for a large excitation at a given frequency it does so as well for a small excitation at the same frequency, so the stability of $\langle A(\mathbf{k}_0\omega_0)\rangle$ implies the existence of a pole in $d^r(\mathbf{k}_0, \omega)$ at $\omega = \omega_0$. In this mode the decay rate vanishes.

The equation for $d^<(\mathbf{k}\omega)$, however, as was demonstrated for $g^<$ (see (1.46) and following remarks), establishes the equilibrium field energy density by balancing the incoherent input (spontaneous emission) into the mode with the rate of decay of the energy in the mode already. But at resonance the photons in the mode do not decay at all. Then there is a buildup of mode energy which is reflected in the divergent result (3.19′) of the evaluation of this quantity.

We can repeat this argument in a slightly different way. The resonance condition balances stimulated emission with absorption. Energy balance, however, requires that absorption equal the sum of spontaneous and stimulated emission. These are inconsistent unless spontaneous emission vanishes. This last would only be the case in something like a truly perfect cavity, where a resonant mode can exist because it does not interact at all with the cavity.

The physical situation, of course, is one of energy balance. As the inversion approaches threshold the energy in the mode in the form of field fluctuations increases due to the increasing lifetime of the excitations. This incoherent[24] energy, rather than a large coherent excitation as we have assumed, serves to modify the inversion to preserve stability. The stable point, however, is defined by the energy balance condition. The modified inversion can never attain the value required for the threshold of a true mode and all excitations have a finite lifetime.

As λ, the unmodified inversion, increases, the stable level of field intensity increases and the spontaneous emission plays a relatively smaller part in the energy balance. That is, the rate of field increment of random phase decreases relative to the field already present. If we can define or measure the phase of the field amplitude at some particular time this phase will experience a random drift due to spontaneous emission, which will become slower as λ increases.

When the intensity is sufficiently large that the rate of phase drift is slow, or excitation lifetime long, compared to the time scale of atomic processes, these processes can be thought of as taking place in the presence of a coherent field whose amplitude is that of the actually incoherent field in the mode. Then we might expect that the calculation of d^r and $d^<$ which has been presented above is still essentially correct, with the obvious modification of giving d^r a small width to correspond to the fact that it is total energy rather than coherent energy which is being balanced, and the replacement of the coherent field intensity $|\langle A(\mathbf{k}_0\omega_0)\rangle|^2$ wherever it occurs with the incoherent expression $(2\pi)^{-1}\int d\omega\langle A(\mathbf{k}\omega)A(-\mathbf{k},-\omega)\rangle = -i(2\pi)^{-1}\int d\omega\, d^<(\mathbf{k}\omega)$. Then the quantity ξ which we introduced ad hoc in (3.14) represents, in that equation, the amount by which the resonance condition fails and in (3.15) the resulting linewidth of electromagnetic excitations. We are assuming that ξ is very small compared to the level widths γ_α so that atomic processes occur rapidly compared to the phase drift and also small compared to $b^r(\omega_0)|\langle A\rangle|^2/\omega_0$ (see (3.14)) so that the stimulated emission dominates the spontaneous.

As a result of these arguments we make the tentative identification, from (3.14) and (3.17),

$$-i\int\frac{d\omega}{2\pi}\,d^<(\omega) = \frac{-a^r(\omega_0)}{b^r(\omega_0)} = \frac{\phi^<(\omega_0)}{16\omega_0^2\xi} \tag{3.22}$$

which is, from (3.12a) (3.12b),

$$\xi = \frac{\phi^<(\omega_0)}{16\omega_0^2}\,\frac{|\gamma_{12}^i|^2}{2\gamma_1\,\gamma_2}\left(1-\frac{pe^{z_0^2}}{\lambda}\right)^{-1} \tag{3.22$'$}$$

This is not a very useful form as it involves dipole matrix elements and level

[24] In this context incoherent refers to that part of the field not described by $\langle A(x)\rangle$, that is, the fluctuating part.

lifetimes which are not well known. A more useful form results from an examination of the power output of the laser. From (A.5), (2.22c), and (3.17) we have

$$P_k = \omega_0 p \phi^<(\omega_0)/4\alpha\xi \tag{3.23}$$

Now according to (3.18) we must know $a^<(\omega)$, $b^<(\omega)$, and $\bar{b}^<(\omega)$ in order to evaluate $\phi^<(\omega_0)$. From (2.16), (2.22), (2.5), and the definition of the effective cavity temperature we used in (2.29) we have

$$\begin{aligned} a^<(\omega_0) &= -2i(2\omega_0'^2/\alpha)[p(e^{\beta\omega_0} - 1)^{-1} + \lambda n_2 e^{-z_0^2}/(n_2 - n_1)] \\ a^<(-\omega_0) &= -2i(2\omega_0'^2/\alpha)[pe^{\beta\omega_0}(e^{\beta\omega_0} - 1)^{-1} + \lambda n_1 e^{-z_0^2}/(n_2 - n_1)] \end{aligned} \tag{3.24a}$$

The computation of $b^{\gtrless}$ and $\bar{b}^{\gtrless}$ is carried out in Appendix IV. The results of interest to us are

$$b^<(\omega_0) = -b^<(-\omega_0) = (2\omega_0'^2/\alpha)i\lambda' \exp(-z_0^2) \tag{3.24b}$$

$$\bar{b}^<(\omega_0) = \bar{b}^<(-w_0) = -(2\omega_0'^2/\alpha)i\lambda' \exp(-z_0^2)(n_2 + n_1)/(n_2 - n_1) \tag{3.24c}$$

Using (2.30a) we can now write (3.23) as

$$\tfrac{1}{2}\xi = \frac{\omega_0(\Delta\omega_c)^2}{\text{Power}}\left[\left(\frac{1}{e^{\beta\omega_0} - 1} + \frac{n_2}{n_2 - n_1}\right) + \frac{\lambda\,|\,a\,|e^{-z_0^2}}{p}\,\frac{n_2 + n_1}{n_2 - n_1}\right] \tag{3.25}$$

If we note from (3.15) or (3.17) that ($\tfrac{1}{2}\xi$) is the half width of the laser output we see that (3.25) is half the Townes formula (*16*) with the $n_2/(n_2 - n_1)$ correction found by Shimoda (*24*).[25,26] The additional correction term which is proportional to the field intensity is small in cases where our treatment is valid. Equation (3.25) should be compared with our below threshold result of the full Townes width in (2.29′).

IV. PARTIALLY COHERENT TREATMENT

Although we believe that the calculation outlined in Section III is essentially correct it is not wholly satisfactory. Aside from the appearance in intermediate steps of $\langle A \rangle$, which is strictly zero, it is difficult to see how one can proceed in a systematic way to compute higher order correlation functions. It is, furthermore, hard to estimate the degree of validity of this approach and out of the question to attempt to study the transition region. In order to avoid some of these difficulties we turn to a reformulation of the laser problem couched, in the limit of vanishing forcing field, entirely in terms of correlation functions. Working in these

[25] Our previous report of this result (*11*) also claimed to be half the Townes width but contains an error. The statements in (*11*) are correct with the amendment: Δf is the *half* width at half-maximum.

[26] This value of half the Townes width has also been obtained by Lamb (*5*) and Lax (*19*). Lax includes the Shimoda correction while Lamb does not.

terms we will ultimately recover the frequency and amplitude conditions (2.43) as well as the expression (3.22) or (3.25) for the linewidth.

Our starting point is (2.34), where we now do not replace the cubic term with its factored form. For convenience in part of what follows we generalize somewhat by adding to the subsidiary Hamiltonian (1.3) the nonlocal term

$$-\mathbf{U}^{\zeta\zeta'}(\mathbf{x}\mathbf{x}')\mathbf{A}(\mathbf{x}')\mathbf{A}(\mathbf{x}) \tag{4.1}$$

whose effect[27] is to change (2.34a) into

$$\begin{aligned}(a^{\zeta\zeta_1}(\mathbf{x}\mathbf{x}_1) - \zeta[U^{\zeta\zeta_1}(\mathbf{x}\mathbf{x}_1) + U^{\zeta_1\zeta}(\mathbf{x}_1\mathbf{x})])\langle A^{\zeta_1}(\mathbf{x}_1)\rangle \\ + b^{\zeta\zeta_1\zeta_2\zeta_3}(\mathbf{x}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3)\langle A^{\zeta_1}(\mathbf{x}_1)A^{\zeta_2}(\mathbf{x}_2)A^{\zeta_3}(\mathbf{x}_3)\rangle = j^{\zeta}(\mathbf{x})\end{aligned} \tag{4.2}$$

Variation with respect to **U** gives, for example,

$$\frac{\delta\langle A^{\zeta}(\mathbf{x})\rangle}{\delta U^{\zeta_1\zeta_2}(\mathbf{x}_1\,\mathbf{x}_2)} = i(\langle A^{\zeta}(\mathbf{x})A^{\zeta_1}(\mathbf{x}_1)A^{\zeta_2}(\mathbf{x}_2)\rangle - \langle A^{\zeta}(\mathbf{x})\rangle\langle A^{\zeta_1}(\mathbf{x}_1)A^{\zeta_2}(\mathbf{x}_2)\rangle) \tag{4.3}$$

U and **j** can be varied independently and both are set to zero after the last variation is performed. The fact that the matter correlation functions which enter into the calculation of **a** and **b** only depend on the "normal" part of the radiation field (cf. (A.8)) is taken into account by specifying that neither **a** nor **b** depends on **j** or **U**.

Varying (4.2) with respect to **j** and setting **j** to zero we find

$$\begin{aligned}(a^{\zeta\zeta_1}(\mathbf{x}\mathbf{x}_1) - \zeta[U^{\zeta\zeta_1}(\mathbf{x}\mathbf{x}_1) + U^{\zeta_1\zeta}(\mathbf{x}_1\mathbf{x})])D^{\zeta_1\zeta'}(\mathbf{x}_1\mathbf{x}') \\ + b^{\zeta\zeta_1\zeta_2\zeta_3}(\mathbf{x}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3)(i\zeta')\langle A^{\zeta_1}(\mathbf{x}_1)A^{\zeta_2}(\mathbf{x}_2)A^{\zeta_3}(\mathbf{x}_3)A^{\zeta'}(\mathbf{x}')\rangle = \delta^{\zeta\zeta'}(\mathbf{x}-\mathbf{x}')\end{aligned} \tag{4.4}$$

where we have noted that $\langle A(\mathbf{x})\rangle = 0$ when $\mathbf{j} = 0$. Equation (4.4) with the definitions and evaluations of **a** and **b** now allows a complete description of the radiation field in terms of correlation functions only. By taking successive derivatives with respect to **U** we can find from (4.4) a hierarchy of equations for the two four, etc. field correlations. Were we able to solve these equations we would presumably find that with an increase of the inversion parameter λ the spectrum of **D** would narrow sharply in one or a few cavity modes. Beyond some threshold region the higher correlation functions would rapidly change from a "thermal" form to one displaying partial coherence.

A. Incoherent Approximation

We shall first analyze (4.4) on the basis of a perturbation expansion in powers of the photon-photon interaction $b^{\zeta\zeta_1\zeta_2\zeta_3}$, which is equivalent to a series of successive truncations of the higher order equations found by functionally dif-

[27] As mentioned in footnote 8, we must generalize (1.4) in order to include such a term. This is easily accomplished and the effects are as stated.

ferentiating (4.4). We shall show that this expansion diverges above threshold and then go on to consider another procedure which we believe to be more correct and which reproduces our earlier results as the expansion does not.

Using the analogue of (4.3) we write the quartic term in (4.4) as

$$-ib^{\zeta\zeta_1\zeta_2\zeta_3}(\mathbf{x}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3)[D^{\zeta_2\zeta_3}(\mathbf{x}_2\mathbf{x}_3)\zeta_3 D^{\zeta_1\zeta'}(\mathbf{x}_1\mathbf{x}') + \delta D^{\zeta_1\zeta'}(\mathbf{x}_1\mathbf{x}')/\delta U^{\zeta_2\zeta_3}(\mathbf{x}_2\mathbf{x}_3)] \quad (4.5)$$

Then, defining $\mathbf{D}^{-1}$ by the analogue of (1.15) and using (1.16) we have

$$\begin{aligned}\{a^{\zeta\zeta_1}(\mathbf{x}\mathbf{x}_1) - \zeta[U^{\zeta\zeta_1}(\mathbf{x}\mathbf{x}_1) + U^{\zeta_1\zeta}(\mathbf{x}_1\mathbf{x})] - ib^{\zeta\zeta_1\zeta_2\zeta_3}(\mathbf{x}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3)D^{\zeta_2\zeta_3}(\mathbf{x}_2\mathbf{x}_3)\zeta_3 \\ + ib^{\zeta\zeta_4\zeta_2\zeta_3}(\mathbf{x}\mathbf{x}_4\mathbf{x}_2\mathbf{x}_3)D^{\zeta_4\zeta_5}(\mathbf{x}_4\mathbf{x}_5)\delta D^{-1\,\zeta_5\zeta_1}(\mathbf{x}_5\mathbf{x}_1)/\delta U^{\zeta_2\zeta_3}(\mathbf{x}_2\mathbf{x}_3)\}D^{\zeta_1\zeta'}(\mathbf{x}_1\mathbf{x}') \quad (4.6) \\ = \delta^{\zeta\zeta'}(\mathbf{x} - \mathbf{x}')\end{aligned}$$

which is an equation for $\mathbf{D}^{-1}$. This equation permits an expansion of $\mathbf{D}^{-1}$ in powers of $\mathbf{b}$ and $\mathbf{D}$. To terms of order $\mathbf{b}^2$, setting $\mathbf{U} = 0$, we find

$$\begin{aligned}D^{-1}(11') = a(11') - ib(1456)D(23)\zeta_3[\delta(41')\delta(52)\delta(63) \\ + \delta(51')\delta(42)\delta(63) + \delta(61')\delta(42)\delta(53)] \\ + b(1523)D(54)b(4678)D(22')\zeta_2'D(33')\zeta_3' \\ \cdot[\delta(61')\delta(72')\delta(83') + \delta(61')\delta(73')\delta(82') \quad (4.7) \\ + \delta(71')\delta(62')\delta(83') + \delta(71')\delta(63')\delta(82') \\ + \delta(81')\delta(62')\delta(73') + \delta(81')\delta(72')\delta(63')]\end{aligned}$$

where 1 denotes $(\mathbf{x}_1, \zeta_1)$ and $\delta(12) \equiv \delta^{\zeta_1\zeta_2}(\mathbf{x}_1 - \mathbf{x}_2)$.

For such an expansion to be meaningful we should be able to compute $\mathbf{D}^{-1}$ to first order in $\mathbf{b}$ and use the resultant $\mathbf{D}$ to find the, presumably small, corrections arising from the second order term. The equation for $\mathbf{D}$ to first order involves, from (4.7),

$$[b(11'23) + b(121'3) + b(1231')]D(23)\zeta_3 \quad (4.8)$$

Now there is time translational invariance in the system since no true oscillation can exist. Then $\mathbf{D}(\omega, \omega') = 2\pi\mathbf{D}(\omega)\delta(\omega - \omega')$. From the representation (3.7a) of $\mathbf{b}$ and (3.4) of $\mathbf{D}$ and the statement (see (A.15)) that $\mathbf{b}(\omega,\omega_1\omega_2\omega_3)$ is only appreciably for $\omega_1 \approx \omega_3$ we see that the second term in (4.8) can make no contribution.

We suppose that we are well above threshold so that the linewidth is small compared to the atomic widths γ_α. Now $\mathbf{b}(\omega\omega_1\omega_2\omega_3)$ has a frequency variation tion on the scale of γ_α, so we may set

$$\int_{-\infty}^{\infty} d\omega_1\, \mathbf{b}(\omega, \omega_1, -\omega_1, \omega)\mathbf{D}(\omega_1) \approx \mathbf{b}(\omega, \omega_0, -\omega_0, \omega) \int_0^{\infty} d\omega_1\, \mathbf{D}(\omega_1) \quad (4.9)$$

in (4.8) where ω_0 is the line center and we have taken ω positive.

Finally we suppose that the intensity is so high that we may neglect d^r relative to $d^>$ or $d^<$, so that in the representation (1.7), (1.8)

$$D^{\zeta\zeta'}\zeta' = \begin{pmatrix} d^r & 0 \\ d^r - d^a & -d^a \end{pmatrix} + \begin{pmatrix} d^< & d^< \\ d^< & d^< \end{pmatrix} \tag{4.10}$$

we can neglect the first matrix relative to the second.

Following these arguments the first order equation for $\mathbf{D}$ is

$$\left[a^{\zeta\zeta_1}(\omega) + 2b^{\zeta\zeta_1}(\omega)(2\pi)^{-1}\int_0^\infty d\omega_1(-i\,d^<(\omega_1))\right]D^{\zeta_1\zeta'}(\omega) = \delta^{\zeta\zeta'} \tag{4.11}$$

where $b^{\zeta\zeta_1}$ is defined in (3.10a).

The usual decomposition (1.54) of (4.11) yields

$$d^r(\omega) = [a^r(\omega) + 2b^r(\omega)\,|\,\Lambda\,|]^{-1} \tag{4.12a}$$

$$\begin{aligned} 2\pi\,|\,\Lambda\,| &\equiv -i\int_0^\infty d\omega\, d^<(\omega) \\ &= i\int_0^\infty d\omega\, d^r(\omega)(a^<(\omega) + 2b^<(\omega)\,|\,\Lambda\,|)\, d^a(\omega). \end{aligned} \tag{4.12b}$$

Using our evaluation (3.12) of a^r and b^r we write (4.12a) as

$$\begin{aligned} 2\omega_0'^2\, d^r(\omega)/\alpha = -\{&\delta + (z_0 - z_0') + 2\lambda\pi^{-1/2}f_1(z_0 + \delta) \\ &+ i(p - \lambda\exp(-z^2)) - 2\lambda'|\,\Lambda\,|\exp(-z^2) \\ &\cdot[\Gamma_1/(\delta + 2i\Gamma_1) + \Gamma_2/(\delta + 2i\Gamma_2)]\}^{-1} \end{aligned} \tag{4.13}$$

where $\delta \equiv z - z_0$.

We have said that the photon line is narrow and centered at $z = z_0$. Then we require

$$z_0 - z_0' + 2\lambda\pi^{-1/2}f_1(z_0) = 0 \tag{4.14a}$$

$$p + (2\lambda'|\,\Lambda\,| - \lambda)\exp(-z_0^2) = \alpha\xi/2\omega_0 \tag{4.14b}$$

where ξ is the full linewidth in frequency units.

Equation (4.14a) fixes z_0, the center frequency of the resonance, and is identical to our previous result (see (2.43)). Equation (4.14b) relates the linewidth to the intensity $|\,\Lambda\,|$ and, in conjunction with (4.12b), determines both these quantities. Note that the stability requirements, $\xi \geqq 0$, puts a lower bound on $|\,\Lambda\,|$. When the linewidth is small compared to the width of the empty-cavity resonance, and for a reasonable level of excitation, $\alpha\xi/2\omega_0$ can be neglected when determining $|\,\Lambda\,|$ from (4.14b). Then comparison with (2.43) shows that we predict half the intensity of oscillation as in the "pure mode" theory.

From (4.12b), using (3.24) and a Lorentzian approximation to (4.13), we may evaluate ξ by relating it to $|\Lambda|$. For comparison with our previous results we use (A.5), (2.22c), and (2.30a) to relate this to the power output. We find

$$\tfrac{1}{2}\xi = \frac{2\omega_0(\Delta\omega_c)^2}{\text{Power}}\left[\left(\frac{1}{e^{\beta\omega_0}-1}+\frac{n_2}{n_2-n_1}\right)+\frac{\lambda'|\Lambda|\exp(-z_0^2)}{p}\frac{n_2+n_1}{n_2-n_1}\right] \tag{4.15}$$

A comparison with (3.25) shows us that we now predict the full Townes width as in our linear theory and in contrast to our "mode theory" prediction of half this amount.

Now that we have discussed the first order evaluation of $\mathbf{D}$ let us proceed to examine the effects of the $\mathbf{b}^2$ terms in (4.7). After eliminating those terms which are not invariant under time translation we have

$$\begin{aligned} D^{-1\,\zeta\zeta_1}(\omega) - \text{first order part} &= b^{\zeta\zeta_5\zeta_2\zeta_3}(\omega\omega_1\omega_2\omega_3)D^{\zeta_5\zeta_4}(\omega_1) \\ &\times D^{\zeta_2\zeta_2'}(\omega_2)\zeta_2' D^{\zeta_3\zeta_3'}(\omega_3)\zeta_3'[b^{\zeta_4\zeta_1\zeta_3'\zeta_2'}(\omega_1, \omega, -\omega_3, -\omega_2) \\ &\quad + b^{\zeta_4\zeta_2'\zeta_3'\zeta_1}(\omega_1, -\omega_2, -\omega_3, \omega)] \end{aligned} \tag{4.16}$$

We again ignore the variation in $\mathbf{b}$ when performing the ω_1, ω_2, ω_3 integrals except that the factor $\delta(\omega - \omega_1 - \omega_2 - \omega_3)$ must be noted. The dominant term is that for which each $D^{\zeta\zeta'}\zeta'$ is replaced by $d^<$. If we write, from ((4.12b), (4.13), (4.14b))

$$d^<(\omega) = i\xi|\Lambda|[(\omega-\omega_0)^2 + \tfrac{1}{4}\xi^2]^{-1} \tag{4.17}$$

we may evaluate (4.16) as

$$-6i\,|b^r(w)|^2\,|\Lambda|^3\,\xi[(w-w_0)^2 + 9\xi^2/4]^{-1}\zeta_1 \tag{4.18}$$

Because of the factor ζ_1 the contribution of (4.18) to $(d^r)^{-1}$ vanishes. However, the inclusion of the contribution of (4.18) to $(D^{-1})^<$ changes (4.12b) into

$$|\Lambda| = (4\omega_0^2\xi)^{-1}[i(a^<(\omega_0) + 2b^<(\omega_0)|\Lambda|) + 2\,|b^r(\omega_0)|^2\,|\Lambda|^3/\xi] \tag{4.19}$$

Then the ratio of the second order contribution to $|\Lambda|$ itself is

$$|\Lambda|_2/|\Lambda| = \tfrac{1}{2}\,|b^r(\omega_0)\Lambda/\omega_0\xi|^2 \tag{4.19$'$}$$

But, from (4.12a), $ib^r(\omega_0)|\Lambda|/\omega_0\xi$ is the ratio of the modification of the linewidth due to the laser field to the residual linewidth, roughly the ratio of stimulated to spontaneous emission. Under our conditions this is a very large number so the first "small correction" to our expansion scheme leads to a contribution to the spontaneous emission into the mode very large compared to the lowest order result.

It is not difficult to find the contribution to $(d^r)^{-1}$ by considering the next most dominant term in (4.16). In the product $\mathbf{D}(\omega_1)\mathbf{D}(\omega_2)\zeta_2\mathbf{D}(\omega_3)\zeta_3$ we replace two of the $D^{\zeta\zeta'}(\omega)\zeta'$ by $d^{<}(\omega)$ and one by the remaining matrix in (4.10). A straightforward computation yields

$$d^r(\omega)^{-1} = a^r(\omega) + 2b^r(\omega)|\Lambda| + |b^r(\omega)|^2|\Lambda|^2(\omega - \omega_0 + 3i\xi)^{-1}\omega_0^{-1} \quad (4.20)$$

We merely note that the ratio of the imaginary part of $d^r(\omega)^{-1}$ at $\omega = \omega_0$ due to the second order term to the contribution of the first order term is $-ib^r(-\omega_0)|\Lambda|/6\omega_0\xi$ which is very large.

Then the perturbation expansion of (4.4) in powers of the two-photon interaction seems to diverge. This is not very surprising. The first approximation this yields for the four field correlation function (cf. (4.4), (4.7)) is quite different from the "pure oscillation" evaluation $\langle A\rangle\langle A\rangle\langle A\rangle\langle A\rangle$. We have argued, however, that the true correlation functions should not differ radically from those for an oscillation. Then our first approximation is very poor, which leads to large corrections in the higher order terms. We remark that the first order part of the above calculation is the small excitation limit of any theory which treats the laser field as acting only to change the average population densities. Such a theory neglects coherence or, what amounts to the same thing, the strong photon-photon interaction. The full Townes linewidth which we obtained above is characteristic of such theories, and we have shown that such theories are not correct when we are sufficiently above threshold.

We can see what is wrong with this approximation in another related way by looking at the electromagnetic field intensity fluctuations implied by our first order expression for the four point correlation function. Introducing positive and negative frequency components (*6*) of the field $A(\mathbf{x})$ we have

$$\langle \mathcal{E}(\mathbf{x})\mathcal{E}(\mathbf{x}')\rangle - \langle \mathcal{E}(\mathbf{x})\rangle\langle \mathcal{E}(\mathbf{x}')\rangle \equiv \langle A^-(\mathbf{x})A^+(\mathbf{x})A^-(\mathbf{x}')A^+(\mathbf{x}')\rangle - \langle A^-(\mathbf{x})A^+(\mathbf{x})\rangle\langle A^-(\mathbf{x}')A^+(\mathbf{x}')\rangle \quad (4.21)$$

The first term of (4.7), compared to (4.4), is equivalent to

$$-\langle \mathbf{A}(\mathbf{x}_1)\mathbf{A}(\mathbf{x}_2)\mathbf{A}(\mathbf{x}_3)\mathbf{A}(\mathbf{x}_4)\rangle = \mathbf{D}(\mathbf{x}_1\mathbf{x}_2)\zeta_2\mathbf{D}(\mathbf{x}_3\mathbf{x}_4)\zeta_4 + \mathbf{D}(\mathbf{x}_1\mathbf{x}_3)\zeta_3\mathbf{D}(\mathbf{x}_2\mathbf{x}_4)\zeta_4 + \mathbf{D}(\mathbf{x}_1\mathbf{x}_4)\zeta_4\mathbf{D}(\mathbf{x}_2\mathbf{x}_3)\zeta_3 \quad (4.22)$$

Setting $x = x'$[28] this is easily evaluated as

$$\langle \mathcal{E}^2(\mathbf{x})\rangle - \langle \mathcal{E}(\mathbf{x})\rangle^2 = \langle \mathcal{E}(\mathbf{x})\rangle^2 \quad (4.22')$$

But the nonlinearity of the laser which sets a preferred value of the field intensity serves to damp intensity fluctuations, so the enormous fluctuations pre-

[28] For $x \neq x'$ Eq. (4.42) leads to Brown-Twiss oscillations in the intensity correlation function.

dicted by the noninteracting photon approximation (4.22) is another indication that this is a very poor starting point indeed.

B. Partially Coherent Approximation

What we would like to do is take the oscillation case as our starting point and do perturbation theory around it. This is not easy to do in a straightforward way since, in our stationary ensemble, any amount of spontaneous emission leads to an infinite value of $\mathbf{D}$ in the presence of a free oscillation. We can get around this problem, however, by considering the case of forced oscillation, due to the presence of a small polarization $\mathbf{j}(\mathbf{x})$ at the resonant frequency. We can compute correlation functions in this case and then take the limit of vanishing driving force. Let us see where this leads us.

The three field correlation function satisfies the identity

$$\begin{aligned}\langle A(1)A(2)A(3)\rangle &= \langle A(1)\rangle\langle A(2)\rangle\langle A(3)\rangle - iD(12)\zeta_2\langle A(3)\rangle \\ &\quad - iD(13)\zeta_3\langle A(2)\rangle - iD(23)\zeta_3\langle A(1)\rangle + \langle \hat{A}(1)\hat{A}(2)\hat{A}(3)\rangle\end{aligned} \tag{4.23}$$

where we use the shorthand introduced in (4.7) and have defined $\hat{A} \equiv A - \langle A\rangle$. Ignoring the final, fully correlated, term we substitute (4.23) into (4.2) to find $(\mathbf{U} \to 0)$

$$\begin{aligned}&a^{\zeta\zeta_1}(\omega)\langle A^{\zeta_1}(\omega)\rangle \\ &\quad + b^{\zeta\zeta_1\zeta_2\zeta_3}(\omega\omega_1\omega_2\omega_3)\{[\langle A^{\zeta_1}(\omega_1)\rangle\langle A^{\zeta_2}(\omega_2)\rangle - iD^{\zeta_1\zeta_2}(\omega_1, -\omega_2)\zeta_2]\langle A^{\zeta_3}(\omega_3)\rangle \qquad (4.24)\\ &\quad + [-\langle A^{\zeta_1}(\omega_1)\rangle\langle A^{\zeta_3}(\omega_3)\rangle - iD^{\zeta_1\zeta_3}(\omega_1, -\omega_3)\zeta_3]\langle A^{\zeta_2}(\omega_2)\rangle \\ &\quad + [\langle A^{\zeta_2}(\omega_2)\rangle\langle A^{\zeta_3}(\omega_3)\rangle - iD^{\zeta_2\zeta_3}(\omega_2, -\omega_3)\zeta_3]\langle A^{\zeta_1}(\omega_1)\rangle\} = j^{\zeta}(\omega)\end{aligned}$$

We shall now, after variations with respect to j have been performed, set j to zero except at the resonant frequenty ω_0. That is,

$$j^{\zeta}(\omega) \to 2\pi j[e^{i\psi}\delta(\omega - \omega_0) + e^{-i\psi}\delta(\omega + \omega_0)] \tag{4.25}$$

where ω_0 is chosen so as to maximize $\langle A(\omega_0)\rangle$.

If we define, as in (4.12b),

$$\begin{aligned}2\pi \mid \Lambda \mid &\equiv -i \int_0^{\infty} d\omega\, d^{<}(\omega, \omega) \\ 2\pi\Lambda^{\pm} &\equiv -i \int_0^{\infty} d\omega\, d^{<}(\pm\omega, \pm\omega \mp 2\omega_0)\end{aligned} \tag{4.26}$$

and assume that we are in an inversion range which allows us to neglect d^r compared to $d^<$ (4.24) can be written

$$[a^r(\omega) + 2b^r(\omega)(|\Lambda| + |\langle A\rangle|^2)]\langle A(\omega)\rangle + \bar{b}^r(\omega)(\Lambda^\pm - \langle A(\omega)\rangle^2)\langle A(-\omega)\rangle = j(\omega) \tag{4.27}$$

or, noting that only $\langle A(\pm\omega_0)\rangle$ is nonvanishing

$$\begin{pmatrix} a^r(\omega_0) + 2b^r(\omega_0)(|\Lambda| + |\langle A\rangle|^2) & \bar{b}^r(\omega_0)(\Lambda^+ - \langle A_+\rangle^2) \\ \bar{b}^r(-\omega_0)(\Lambda^- - \langle A_-\rangle^2) & a^r(-\omega_0) + 2b^r(-\omega_0)(|\Lambda| + |\langle A\rangle|^2) \end{pmatrix} \cdot \begin{pmatrix} \langle A(\omega_0)\rangle \\ \langle A(-\omega_0)\rangle \end{pmatrix} = j\begin{pmatrix} e^{i\psi} \\ e^{-i\psi} \end{pmatrix} \tag{4.27'}$$

where we have factored out the frequency δ functions.

It will be instructive to write the equation for $\mathbf{D}$ at this point. From (4.24) this is

$$\begin{aligned} &a^{\zeta\zeta_1}(\omega)D^{\zeta_1\zeta'}(\omega, \omega') + b^{\zeta\zeta_1\zeta_2\zeta_3}(\omega\omega_1\omega_2\omega_3)\{[\langle A^{\zeta_1}(\omega_1)\rangle\langle A^{\zeta_2}(\omega_2)\rangle \\ &- iD^{\zeta_1\zeta_2}(\omega_1, -\omega_2)\zeta_2]D^{\zeta_3\zeta'}(\omega_3\omega') + [\langle A^{\zeta_1}(\omega_1)\rangle\langle A^{\zeta_3}(\omega_3)\rangle \\ &- iD^{\zeta_1\zeta_3}(\omega_1, -\omega_3)\zeta_3]D^{\zeta_2\zeta'}(\omega_2\omega') + [\langle A^{\zeta_2}(\omega_2)\rangle\langle A^{\zeta_3}(\omega_3)\rangle \\ &- iD^{\zeta_2\zeta_3}(\omega_2, -\omega_3)\zeta_3]D^{\zeta_1\zeta'}(\omega_1\omega') - i\langle A^{\zeta_3}(\omega_3)\rangle\delta D^{\zeta_1\zeta_2}(\omega_1, -\omega_2)\zeta_2/\delta_j^{\zeta'}(\omega') \\ &- i\langle A^{\zeta_2}(\omega_2)\rangle\delta D^{\zeta_1\zeta_3}(\omega_1, -\omega_3)\zeta_3/\delta_j^{\zeta'}(\omega') \\ &- i\langle A^{\zeta_1}(\omega_1)\rangle\delta D^{\zeta_2\zeta_3}(\omega_2, -\omega_3)\zeta_3/\delta_j^{\zeta'}(\omega')\} = 2\pi\delta^{\zeta\zeta'}(\omega - \omega') \end{aligned} \tag{4.28}$$

It is crucial here to examine the $\delta\mathbf{D}/\delta\mathbf{j}$ terms closely. By an argument similar to that leading to (3.21) we can write

$$\langle\mathbf{A}(\omega_3)\rangle\frac{\delta\mathbf{D}(\omega_1, -\omega_2)}{\delta\mathbf{j}(\omega')} = \frac{(\omega_1 + \omega_2)\mathbf{D}(\omega_1, -\omega_2)\langle\mathbf{A}(\omega_3)\rangle}{\omega'\mathbf{j}(\omega')} \tag{4.29a}$$

which is small in the important range in (4.28), $\omega_1 + \omega_2 \approx 0$. On the other hand

$$\begin{aligned} \langle\mathbf{A}(\omega_2)\rangle\frac{\delta\mathbf{D}(\omega_1, -\omega_3)}{\delta\mathbf{j}(\omega')} &= \frac{\langle\mathbf{A}(\omega_2)\rangle(\omega_1 + \omega_3)\mathbf{D}(\omega_1, -\omega_3)}{\omega'\mathbf{j}(\omega')} \\ &= \mathbf{D}(\omega_1, -\omega_3)\mathbf{D}(\omega_2, \omega')(\omega_1 + \omega_3)/\omega_2 \end{aligned} \tag{4.29b}$$

But in the limit of (4.25) $\omega_1 + \omega_3 = 2\omega_0$ and $\omega_2 \approx -\omega_0$. Then (4.29b) cannot be neglected and, in fact, changes the sign of $\mathbf{D}(\omega_1, -\omega_3)$ in the coefficient of $\mathbf{D}(\omega_2, \omega')$ in (4.28). Using (4.26) and keeping the important terms in (4.28) we have

$$\begin{aligned}[a^{\zeta\zeta_1}(\omega) + 2b^{\zeta\zeta_1}(\omega)(|\Lambda| + |\langle A\rangle|^2)]D^{\zeta_1\zeta'}(\omega_1\omega') \\ - \bar{b}^{\zeta\zeta_1}(\omega)(\Lambda^{\pm}(\omega) - \langle A(\omega)\rangle^2)D^{\zeta_1\zeta'}(\omega - 2\omega_0, \omega') = 2\pi\delta^{\zeta\zeta'}(\omega - \omega')\end{aligned} \tag{4.30}$$

Then d^r satisfies, factoring out the frequency δ function,

$$\begin{pmatrix} a^r(\omega) + 2b^r(\omega)(|\Lambda| + |\langle A\rangle|^2) & -\bar{b}^r(\omega)(\Lambda^+ - \langle A_+\rangle^2) \\ -\bar{b}^r(\omega_-)(\Lambda^- - \langle A_-\rangle^2) & a^r(\omega_-) + 2b^r(\omega_-)(|\Lambda| + |\langle A\rangle|^2)\end{pmatrix} \cdot (\mathbf{D}^r(\omega)) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{4.31}$$

where we use the shorthand $(\mathbf{D}^r(\omega))$ defined in (3.11b).

A comparison of (4.31) with (4.27′) now shows that if $\langle A(\omega_0)\rangle$ can be finite when $j \to 0$, then $(\mathbf{D}^r(\omega))$ has a pole at $\omega = \omega_0$. The same determinant vanishes for both these situations. This is in accord with our exact result and is thus a very satisfactory feature in this approximation. Furthermore, if the spontaneous emission into the mode $\phi^<(\omega_0)$ were zero, we would recover our original mode calculation either from (4.27′) or (4.31), when $j \to 0$. That is, the determinant would be required to be zero either to allow $\langle A\rangle$ not to vanish or to allow $|\Lambda|$ not to vanish, one of the two having to be finite for stability. This is again very satisfactory since spontaneous emission alone is responsible for the failure of the mode theory.

We shall demonstrate a solution to the set of equations (4.27′), (4.30). It is not difficult to show that it is the only solution. We write (4.27′) as

$$\begin{pmatrix}\langle A(\omega_0)\rangle \\ \langle A(-\omega_0)\rangle\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}(\mathbf{D}^r(\omega_0))\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} j \begin{pmatrix}e^{i\psi}\\ e^{-i\psi}\end{pmatrix} \tag{4.32}$$

where $(\mathbf{d}^r)$ can be read off from (4.31). If we assume that

$$(|\Lambda| + |\langle A\rangle|^2)^2 = (\Lambda^+ - \langle A_+\rangle^2)(\Lambda^- - \langle A_-\rangle^2) \tag{4.33}$$

and insist that $(\mathbf{D}^r)$ corresponds to an intense, stable, narrow mode at ω_0 we must set

$$a^r(-\omega_0) + b^r(-\omega_0)(|\Lambda| + |\langle A\rangle|^2) = i\omega_0\xi \tag{4.34}$$

which leads to

$$(\mathbf{D}^r(\omega)) = -\frac{1}{4\omega_0(\omega - \omega_0 + \tfrac{1}{2}i\xi)}\frac{1}{|\Lambda| + |\langle A\rangle|^2} \cdot \begin{pmatrix} |\Lambda| + |\langle A\rangle^2 & -\Lambda^+ + \langle A_+\rangle^2 \\ \Lambda^- - \langle A_-\rangle^2 & -|\Lambda| - |\langle A\rangle|^2\end{pmatrix} \tag{4.35}$$

Note that (4.34) predicts the same frequency as in mode theory and, when ξ is sufficiently small, gives for the sum of coherent and incoherent intensity the same value as the mode theory gave for coherent intensity (cf. 3.14). To check the consistency of this solution we compute

$$(\mathbf{D}^{<}(\omega)) = \frac{i\phi^{<}(\omega_0)}{16\omega_0^2[(\omega-\omega_0)^2+\frac{1}{4}\xi^2]}\frac{1}{|\Lambda|+|\langle A\rangle|^2} \cdot \begin{pmatrix} |\Lambda|+\langle A\rangle|^2 & \Lambda^+-\langle A_+\rangle^2 \\ \Lambda^- - \langle A_-\rangle^2 & |\Lambda|+|\langle A\rangle|^2 \end{pmatrix} \tag{4.36}$$

where $i\phi^{<}(\omega_0)$ is as defined in (3.18) with $|\langle A\rangle|^2 \to |\Lambda| + |\langle A\rangle|^2$. Comparison with the definition of Λ, (4.26), gives the consistency conditions

$$|\Lambda| = \phi^{<}(\omega_0)/16\omega_0^2\xi \tag{4.37a}$$

$$\Lambda^{\pm} = |\Lambda|\frac{\Lambda^{\pm}-\langle A_{\pm}\rangle^2}{|\Lambda|+|\langle A\rangle|^2} = -|\Lambda|\frac{\langle A_{\pm}\rangle^2}{|\langle A\rangle|^2} \tag{4.37b}$$

Equation (4.37a) is the same as (3.22) and thus leads to the same evaluation of the linewidth we found in the mode theory. Equation (4.37b) demonstrates the validity of (4.33) which was the basic assumption we made to get to (4.35). We can now set j to zero. $\langle A\rangle$ goes smoothly to zero and the resulting equations (4.33), (4.34), (4.35), (4.36) remain a consistent set leading to the mode theory frequency and intensity and half the Townes linewidth.

If we consider (4.30) when $j = \langle A\rangle = 0$ we regain the result of our previous expansion procedure (4.11) and all its first order predictions by setting $\Lambda^{\pm} = 0$. This is consistent with (4.37b) but is not the smooth limit of an oscillation theory as $j \to 0$, which is just another way of understanding why corrections to such a theory are large.

It might be argued, however, that the theory we have just outlined is not satisfactory since it requires that, when $j \to 0$, $\Lambda^{\pm}$ does not vanish, (4.33), though it is not a time translationally invariant quantity. (See (4.26).) There are two possible ways to argue that this is not a fatal objection. First, as in the question of whether or not $\langle A(\omega)\rangle$ is zero, the argument of time invariance is not sufficient. $\langle A(\omega_0)\rangle$ could be nonzero if $\phi^{<}(\omega_0) = 0$, that is, a field started with a given phase could retain that phase as long as there was no spontaneous emission to cause that phase to drift. Likewise we cannot insist that $\Lambda^{\pm} = 0$ until we examine the equation for the four field correlation function and show that the assumption of nonvanishing $\Lambda^{\pm}$ leads to an inconsistency. Even if we were forced to conclude from such a calculation that $\Lambda^{\pm} = 0$ we might still justify the use of this calculation of $\mathbf{D}$. Just as our original mode calculation gave us expressions in terms of $|\langle A\rangle|^2$ which we could later identify as $-i\int d^{<}(\omega, \omega)\, d\omega$, so in this calculation we may think of $\Lambda^+\Lambda^-$ as being representative of a correlation function $\langle A(\omega)A(2\omega_0-\omega)\cdot$

$A(-\omega)A(\omega - 2\omega_0)\rangle$ which has a large narrow resonance at $\omega = \omega_0$ and which would enter into a more accurate calculation. Such a calculation might retain the final term in (4.23) or might treat (4.4) or (4.6) in a more accurate way than we have done. Justification of our procedure along these lines remains to be carried out.

It is interesting to look at the correlation functions in the time domain which are given by our evaluation of **D**. From (4.36) with $j = 0$ we can write

$$\langle A(\mathbf{r}t)A(\mathbf{r}'t')\rangle = e^{-\xi|t-t'|/2}(h(\mathbf{r})e^{-i\omega_0 t} + h^*(\mathbf{r})e^{i\omega_0 t})(h(\mathbf{r}')e^{-i\omega_0 t'} + h^*(\mathbf{r}')e^{i\omega_0 t'}) \tag{4.38}$$

where $h(\mathbf{r}) = |\Lambda|^{1/2} e^{i(\mathbf{k}_0\cdot\mathbf{r}+\phi)}$ and ϕ is an arbitrary phase.[29] For small time differences this looks like the product of field expectation values at the two field points.

It is also possible to consider intensity fluctuations. Comparing (4.4) with (4.30) we see that our calculation above is equivalent to the replacement

$$\begin{aligned}\langle A^-(\mathbf{x}_1)A^-(\mathbf{x}_2)A^+(\mathbf{x}_3)A^+(\mathbf{x}_4)\rangle &\approx \langle A^-(\mathbf{x}_1)A^+(\mathbf{x}_3)\rangle\langle A^-(\mathbf{x}_2)A^+(\mathbf{x}_4)\rangle \\ &+ \langle A^-(\mathbf{x}_1)A^+(\mathbf{x}_4)\rangle\langle A^-(\mathbf{x}_2)A^+(\mathbf{x}_3)\rangle - \langle A^-(\mathbf{x}_1)A^-(\mathbf{x}_2)\rangle\langle A^+(\mathbf{x}_3)A^+(\mathbf{x}_4)\rangle\end{aligned} \tag{4.39}$$

where $(\pm)$ again refer to positive and negative frequency parts. From (4.36) this is

$$\begin{aligned}\langle A^-(\mathbf{r}_1t_1)A^-(\mathbf{r}_2t_2)A^+(\mathbf{r}_3t_3)A^+(\mathbf{r}_4t_4)\rangle &\approx h^*(\mathbf{r}_1)h^*(\mathbf{r}_2)h(\mathbf{r}_3)h(\mathbf{r}_4) \\ &\times e^{i\omega_0(t_1+t_2-t_3-t_4)}\{e^{-\xi(|t_1-t_3|+|t_2-t_4|)/2} \\ &+ e^{-\xi(|t_1-t_4|+|t_2-t_3|)/2} - e^{-\xi(|t_1-t_2|+|t_3-t_4|)/2}\}\end{aligned} \tag{4.39$'$}$$

When all the times are close to one another this again looks like the product of the appropriate parts of field expectation values.[30] Furthermore, if we are computing intensity fluctuations, by setting $(\mathbf{r}_1t_1) = (\mathbf{r}_3t_3)$, $(\mathbf{r}_2t_2) = (\mathbf{r}_4t_4)$ this becomes

$$\langle \mathcal{E}(\mathbf{r}_1t_1)\mathcal{E}(\mathbf{r}_2t_2)\rangle = h^*(\mathbf{r}_1)h(\mathbf{r}_1)h^*(\mathbf{r}_2)h(\mathbf{r}_2) = \langle \mathcal{E}(\mathbf{r}_1t_1)\rangle\langle \mathcal{E}(\mathbf{r}_2t_2)\rangle \tag{4.39$''$}$$

so intensity fluctuations vanish identically even though the four point function is not precisely expressible as the product of four fields.

Then we may say that the approximation we have been led to consists in replacing a four field correlation function by an almost coherent expression which has no intensity fluctuations, and the two field correlation function we

[29] ϕ is related to the phase ψ in (4.25) by $\phi = \psi + \frac{1}{2}\pi$. This is relevant if we think of the phase as being defined by an actual external field which adiabatically goes to zero.

[30] In ref. *6*, Eqs. (4.38) and (4.39) above would be the conditions for approximate second order coherence of the laser light signal.

then calculate is almost coherent as well. The degree of coherence we find is the maximum permitted taking spontaneous emission induced phase drift into account. Presumably a calculation of the remainder of the four field correlation would give intensity fluctuations as well. We also expect that higher correlation functions will appear in an almost factorable form.

CONCLUSION

We have discussed the properties of the laser oscillator in three different ways. First we asked for the circumstances under which a pure mode of the field can exist. This led to well defined values for the threshold and for the mode frequency and intensity above threshold. Because of the random nature of spontaneous emission, however, leading to random phase drifts, no pure mode of the field can exist, and this fact manifested itself in the infinite value we found for the incoherent energy in a system supporting a true mode. It was possible to patch up this theory to give finite incoherent energy and a linewidth but fluctuations are not considered in a consistent manner and the presence of a finite value for the mode intensity in intermediate steps is not satisfactory. Our second attempt dispensed with the appearance of a coherent mode and considered fluctuations consistently but ignored the possibility of partial coherence in the laser field. The unsatisfactory nature of such an approximation manifested itself in the dominance of higher order corrections over the lowest order terms and the presumed divergence of the perturbation theory. As noted by other authors (*5*, *19*) such an incoherent theory gives rise to an intensity-linewidth relation differing from that of the coherent theory by a factor of two. We showed as well that the intensity predicted also differs by a factor of two, for a given inversion.

Our final discussion takes account both of fluctuations and coherence, though intensity fluctuations do not yet appear to the order we have considered. Its predictions are almost identical to those of the "patched up" pure mode theory. In particular it predicts the half-Townes linewidth appropriate to a coherent theory and its evaluations of the first two field correlation functions, apart from small phase drift corrections, are those of a fully coherent field.

APPENDIX I. POWER ABSORBED BY THE CAVITY

We write the Hamiltonian (1.1) in the form

$$H(t) = \int d\mathbf{r}[H_M(\mathbf{r}t) - \mathbf{A}(\mathbf{r}t)\cdot\mathbf{J}(\mathbf{r}t) + H_{\mathrm{EM}}(\mathbf{r}t)]$$

The rate of change of electromagnetic field energy is given by

$$\begin{aligned} i(\partial/\partial t)\int d\mathbf{r}H_{\mathrm{EM}}(rt) &= \int d\mathbf{r}[H_{\mathrm{EM}}(\mathbf{r}t), H] \\ &= -\int d\mathbf{r}\, d\mathbf{r}'[H_{\mathrm{EM}}(\mathbf{r}t), \mathbf{J}(\mathbf{r}'t)\cdot\mathbf{A}(\mathbf{r}'t)] = \int d\mathbf{r}'(\mathbf{J}(\mathbf{r}'t)\cdot i\partial\mathbf{A}(\mathbf{r}'t)/\partial t) \end{aligned}$$

But $\Box \mathbf{A}(\mathbf{r}t) = \mathbf{J}(\mathbf{r}t)$ so

$$\left\langle \frac{\partial}{\partial t} H_{\mathrm{EM}}(t) \right\rangle = \int d\mathbf{r} \lim_{t' \to t} \lim_{\mathbf{r}' \to \mathbf{r}} \frac{\partial}{\partial t'} \Box \langle \mathbf{A}(\mathbf{r}t) \cdot \mathbf{A}(\mathbf{r}'t') \rangle$$
$$= \int d\mathbf{r} \lim_{t' \to t} \lim_{\mathbf{r}' \to \mathbf{r}} \left(-i \frac{\partial}{\partial t'} \right) \Box\, d^{>}(\mathbf{r}t, \mathbf{r}'t') + \text{coherent term} \tag{A.1}$$

Now from (1.54c)

$$\Box\, d^{>}(\mathbf{r}t, \mathbf{r}'t') = p^{r}(\mathbf{r}t, \mathbf{r}_1 t_1)\, d^{>}(\mathbf{r}_1 t_1, \mathbf{r}'t') + p^{>}(\mathbf{r}t, \mathbf{r}_1 t_1)\, d^{a}(\mathbf{r}_1 t_1, \mathbf{r}'t')$$

so

$$\langle \partial H_{\mathrm{EM}}(t)/\partial t \rangle$$
$$= -i \int d\mathbf{r} \{ p^{r}(\mathbf{r}t, \mathbf{r}_1\, t_1) \partial\, d^{>}(\mathbf{r}_1\, t_1, \mathbf{r}t)/\partial t + p^{>}(\mathbf{r}t, \mathbf{r}_1\, t_1) \partial\, d^{a}(\mathbf{r}_1\, t_1, \mathbf{r}t)/\partial t \} \tag{A.2}$$

In the steady state this is zero but we can discuss the energy exchange with a given element of the matter system by inserting its contribution to the polarization operator in (A.2). Noting that the cavity system is translationally invariant in space and time and using the form (3.4) for **D** we have

$$\langle \partial H_{\mathrm{EM}}(t)/\partial t \rangle_{\mathrm{CAV}} = (2\pi)^{-5} \int_{-\infty}^{\infty} d\omega' \int d^3\mathbf{k}\, d\omega e^{-i(\omega-\omega')t} \omega' \int d\mathbf{r}$$
$$\times \{ p_c^{\,r}(\mathbf{k}\omega)\, d^{>}(\mathbf{k}\omega, \mathbf{k}\omega') + p_c^{\,>}(\mathbf{k}\omega)\, d^{a}(\mathbf{k}\omega, \mathbf{k}\omega') \} \tag{A.3}$$

If we take the time average of (A.3) only $\omega' = \omega$ appears. Then we may use the relations, following from (1.7) and (1.54) in the absence of active atoms,

$$p_c^{\,r}(\mathbf{k}\omega) = p_c^{\,a}(-k, -\omega), \qquad p_c^{\,>}(\mathbf{k}\omega) = p_c^{\,<}(-\mathbf{k}, -\omega)$$
$$d^{r}(\mathbf{k}\omega, \mathbf{k}'\omega') = d^{a}(-\mathbf{k}', -\omega'; -\mathbf{k}, -\omega),$$
$$d^{>}(\mathbf{k}\omega, \mathbf{k}'\omega') = d^{<}(-\mathbf{k}', -\omega'; -\mathbf{k}, -\omega)$$

to write

$$\langle \partial H_{\mathrm{EM}}(t)/\partial t \rangle_{\mathrm{CAV,AV}} = (2\pi)^{-4} \int d^3\mathbf{k} \int d\mathbf{r}$$
$$\cdot \int_0^{\infty} d\omega\, \omega [(p_c^{\,>}(\mathbf{k}\omega) - p_c^{\,<}(\mathbf{k}\omega))\, d^{<}(\mathbf{k}\omega) - p_c^{\,<}(\mathbf{k}\omega)(d^{>}(\mathbf{k}\omega) - d^{<}(\mathbf{k}\omega))] \tag{A.4}$$

The first term in (A.4) is proportional to the field intensity so we may take it as the energy absorbed by the cavity. The second is proportional to $p_c^{\,<}(\mathbf{k}\omega)$ and is the spontaneous emission of the cavity material. We may then restrict ourselves to the first term to find

$$(\mathrm{Vol.})^{-1}\ \mathrm{Power} = -\int \frac{d^3\mathbf{k}}{(2\pi)^3} \int_0^{\infty} \frac{d\omega}{2\pi}\, i\omega^2 \bar{p}_c(\mathbf{k}\omega)\, d^{<}(\mathbf{k}\omega) \tag{A.5}$$

where we have used (1.58) and the negative sign arises because power removed from the field is considered positive.

If the ω variation in $\bar{p}_c(\mathbf{k}\omega)$ is slow compared to that in $d^<(\mathbf{k}\omega)$ we may write approximately, for the mode $\mathbf{k}$

$$P_{\mathbf{k}} = -(\omega_0(\mathbf{k})/Q_{\mathbf{k}}) \times 2(2\pi)^{-1} \int_0^\infty d\omega\, i\omega^2\, d^<(\mathbf{k}\omega) = -\omega_0\langle \mathcal{E}(\mathbf{k})\rangle/Q_{\mathbf{k}} \quad (\mathrm{A.6})$$

where $\langle \mathcal{E}_{\mathbf{k}}\rangle$ is the energy in the mode $\mathbf{k}$ and we have used (2.20b). This is a well known result and serves to verify (A.5).

APPENDIX II. EXPANSION OF $G_{\alpha\beta}$

We expand $G_{\alpha\beta}^{\zeta\zeta}(\mathbf{xx})$ according to the prescription in (2.33). The iterative expansion of the equation for $\mathbf{M}$, (1.20), is, to order $(\gamma_{\alpha\beta}^i)^3$,

$$\begin{aligned} M_{\alpha\beta}^{\zeta\zeta'}(\mathbf{xx}') = &-\gamma_{\alpha\beta}^i\langle A_i^{\zeta}(\mathbf{x})\rangle - i\zeta\gamma_{\alpha\sigma}^i G_{\sigma\lambda}^{\zeta\zeta'}(\mathbf{xx}')\gamma_{\lambda\beta}^j\delta\langle A_j^{\zeta'}(\mathbf{x}')\rangle/\delta j_i^{\zeta}(\mathbf{x}) \\ &+ \zeta\zeta_1\gamma_{\alpha\sigma}^i G_{\sigma\lambda}^{\zeta\zeta_1}(\mathbf{xx}_1)\gamma_{\lambda\tau}^j G_{\tau\phi}^{\zeta_1\zeta'}(\mathbf{x}_1\mathbf{x}')\gamma_{\phi\beta}^l\delta^2\langle A_l^{\zeta'}(\mathbf{x}')\rangle/\delta j_i^{\zeta}(\mathbf{x})\delta j_j^{\zeta_1}(\mathbf{x}_1) \end{aligned} \quad (\mathrm{A.7})$$

To express (A.7) in terms of $\hat{\mathbf{G}}$ rather than $\mathbf{G}$ we expand the second term in (A.7) to find

$$\begin{aligned} M_{\alpha\beta}(\mathbf{xx}') = &-\gamma_{\alpha\beta}^i\langle A_i(\mathbf{x})\rangle\delta(\mathbf{x}-\mathbf{x}') - i\gamma_{\alpha\sigma}^i\hat{G}_\sigma(\mathbf{xx}')\gamma_{\sigma\beta}^j D_{ij}(\mathbf{xx}')^{\zeta'} \\ &+ i\gamma_{\alpha\sigma}^i\hat{G}_\sigma(\mathbf{xx}_1)\gamma_{\sigma\lambda}^j\hat{G}_\lambda(\mathbf{x}_1\mathbf{x}')\gamma_{\lambda\beta}^l[\langle A_j(\mathbf{x}_1)\rangle D_{il}(\mathbf{xx}')^{\zeta'} \\ &- i\zeta\zeta'\delta^2\langle A_j(\mathbf{x}_1)\rangle/\delta j_i(\mathbf{x})\delta j_l(\mathbf{x}')] \end{aligned} \quad (\mathrm{A.7}')$$

where we have suppressed ζ indices for clarity and where we have used (1.8a) and (1.21).

For $M^0{}_{\alpha\alpha}$ we take the part of M

$$M_{\alpha\alpha}^{0\zeta\zeta'}(\mathbf{xx}') \equiv -i\,|\,\gamma_{\alpha\sigma}^i\,|^2\,\hat{G}_\sigma^{\zeta\zeta'}(\mathbf{xx}')D_{ii}^{\zeta\zeta'}(\mathbf{xx}')\zeta' \quad (\mathrm{A.8})$$

which we have previously analyzed, (1.31 ff), but specifically restrict $\mathbf{D}$, the photon density, to that part which is not associated with the resonant modes. The remainder, M^1, is then the full expression (A.7′) minus M_0. Then, with (2.33b), we have

$$\begin{aligned} i\zeta\gamma_{\alpha\beta}^i G_{\beta\alpha}(\mathbf{xx}) = i\zeta\gamma_{\alpha\beta}^i\{ &- \hat{G}_\beta(\mathbf{xx}_1)\hat{G}_\alpha(\mathbf{x}_1\mathbf{x})\gamma_{\beta\alpha}^j\langle A_j(\mathbf{x}_1)\rangle \\ &+ \hat{G}_\beta(\mathbf{xx}_1)\gamma_{\beta\sigma}^j\hat{G}_\sigma(\mathbf{x}_1\mathbf{x}_2)\gamma_{\sigma\alpha}^l\hat{G}_\alpha(\mathbf{x}_2\mathbf{x})[\langle A_j(\mathbf{x}_1)\rangle\langle A_l(\mathbf{x}_2)\rangle - iD_{jl}(x_1x_2)\zeta_2] \\ &- \hat{G}_\beta(\mathbf{xx}_1)\gamma_{\beta\sigma}^j\hat{G}_\sigma(\mathbf{x}_1\mathbf{x}_2)\gamma_{\sigma\lambda}^k\hat{G}_\lambda(\mathbf{x}_2\mathbf{x}_3)\gamma_{\lambda\alpha}^l\hat{G}_\alpha(\mathbf{x}_3\mathbf{x})[\langle A_j(\mathbf{x}_1)\rangle\langle A_k(\mathbf{x}_2)\rangle\langle A_l(\mathbf{x}_3)\rangle \\ &- i\langle A_j(\mathbf{x}_1)\rangle D_{kl}(\mathbf{x}_2\mathbf{x}_3)\zeta_3 - i\langle A_k(\mathbf{x}_2)\rangle D_{jl}(x_1x_3)\zeta_3 - i\langle A_l(\mathbf{x}_3)\rangle D_{jk}(\mathbf{x}_1\mathbf{x}_2)\zeta_2 \\ &- \zeta_1\zeta_2\delta^2\langle A_l(\mathbf{x}_3)\rangle/\delta j_j(\mathbf{x}_1)\delta j_k(\mathbf{x}_2)]\} \end{aligned} \quad (\mathrm{A.9})$$

Now from (1.7) and (1.8)

$$\langle A_j^{\zeta_1}(\mathbf{x}_1)\rangle\langle A_k^{\zeta_2}(\mathbf{x}_2)\rangle - iD_{jk}^{\zeta_1\zeta_2}(\mathbf{x}_1\mathbf{x}_2)\zeta_2 = \langle A_j^{\zeta_1}(\mathbf{x}_1)A_k^{\zeta_2}(\mathbf{x}_2)\rangle$$

and the last term in (A.9) is just $\langle A_j^{\zeta_1}(\mathbf{x}_1)A_k^{\zeta_2}(\mathbf{x}_2)A_l^{\zeta_3}(\mathbf{x}_3)\rangle$. Furthermore, the term involving $\langle\mathbf{AA}\rangle$ vanishes, since it is not invariant under charge conjugation $\gamma^i_{\alpha\beta} \to -\gamma^i_{\alpha\beta}$ and would lead to a polarization even when $\langle\mathbf{A}\rangle = 0$. Then we rewrite (A.9) as

$$\begin{aligned} i\zeta\gamma^i_{\alpha\beta}G^{\zeta\zeta}_{\beta\alpha}(\mathbf{xx}) &= -i\zeta \mid \gamma^i_{\alpha\beta}\mid^2 \hat{G}^{\zeta\zeta_1}_{\beta}(\mathbf{xx}_1)\hat{G}^{\zeta_1\zeta}_{\alpha}(\mathbf{x}_1\mathbf{x})\langle A_i^{\zeta_1}(\mathbf{x}_1)\rangle \\ &\quad - i\zeta\gamma^i_{\alpha\beta}\gamma^j_{\beta\sigma}\gamma^k_{\sigma\lambda}\gamma^l_{\lambda\alpha}\hat{G}^{\zeta\zeta_1}_{\beta}(\mathbf{xx}_1)\hat{G}^{\zeta_1\zeta_2}_{\sigma}(\mathbf{x}_1\mathbf{x}_2)\hat{G}^{\zeta_2\zeta_3}_{\lambda}(\mathbf{x}_2\mathbf{x}_3)\hat{G}^{\zeta_3\zeta}_{\alpha}(\mathbf{x}_3\mathbf{x}) \quad (A.10) \\ &\quad \times \langle A_j^{\zeta_1}(\mathbf{x}_1)A_k^{\zeta_2}(\mathbf{x}_2)A_l^{\zeta_3}(\mathbf{x}_3)\rangle \end{aligned}$$

APPENDIX III. EVALUATION OF b^r

Writing

$$\begin{aligned} b^r(\mathbf{xx}_1\,\mathbf{x}_2\,\mathbf{x}_3) &\equiv (2\pi)^{-16}\int d\mathbf{k}\,d\mathbf{k}'\,d\mathbf{k}''\,d\mathbf{k}'''\,d\omega\,d\omega'\,d\omega''\,d\omega''' \\ &\qquad \cdot b^r(\mathbf{k}\omega, \mathbf{k}'\omega', \mathbf{k}''\omega'', \mathbf{k}'''\omega''') \qquad (A.11) \\ &\quad \times \exp i[\mathbf{k}\cdot\mathbf{r} - \mathbf{k}'\cdot\mathbf{r}_1 - \mathbf{k}''\cdot\mathbf{r}_2 - \mathbf{k}'''\cdot\mathbf{r}_3 - \omega t + \omega' t_1 + \omega'' t_2 + \omega''' t_3)] \end{aligned}$$

we may use (2.1) and (2.2′) to express the first term in (2.38) as

$$\begin{aligned} b^r_{ijkl}(\mathbf{k}\omega, \mathbf{k}'\omega', \mathbf{k}''\omega'', \mathbf{k}'''\omega''')_1 &= i\gamma^i_{\alpha\beta}\,\gamma^j_{\beta\sigma}\,\gamma^k_{\sigma\lambda}\,\gamma^l_{\lambda\alpha}(2\pi)\delta(\omega - \omega' - \omega'' - \omega''') \\ &\times \int d^3\mathbf{K}\delta(\mathbf{k} - \mathbf{k}' - \mathbf{k}'' - \mathbf{k}''' - \mathbf{K})(2\pi)^{-4} \\ &\qquad \cdot \int d^3\mathbf{k}_1\,d\omega_1\,g_\beta{}^r(\mathbf{k}_1\,\omega_1)g_\sigma{}^r(\mathbf{k}_1 - \mathbf{k}', \omega_1 - \omega') \qquad (A.12) \\ &\times g_\lambda{}^r(\mathbf{k}_1 - \mathbf{k}' - \mathbf{k}'', \omega_1 - \omega' - \omega'')n_\alpha(\mathbf{k}_1 - \mathbf{k} + \tfrac{1}{2}\mathbf{K}, \mathbf{K}) \\ &\times [g_\alpha{}^r(\mathbf{k}_1 - \mathbf{k} + \tfrac{1}{2}\mathbf{K}, \omega_1 - \omega) - g_\alpha{}^a(\mathbf{k}_1 - \mathbf{k} + \tfrac{1}{2}\mathbf{K}, \omega_1 - \omega)] \end{aligned}$$

We recognize that the product of four retarded functions gives no contribution. Further, only very small values of $\mathbf{K}$ are allowed, corresponding to slow spatial variations of the medium. Then we may ignore $\mathbf{K}$ in the argument of g^r and also in the first argument of n. Finally we are only interested in b^r for $\mathbf{k}'$, $\mathbf{k}''$, $\mathbf{k}'''$ photon momenta, and these are small compared to the range of $n_\alpha(\mathbf{k}, \mathbf{K})$ in its first argument and will be dropped there. Then using (2.9) the integrand in (A.12) is

$$\begin{aligned} -(2\pi)^{-4}\int d^3\mathbf{k}_1\,d\omega_1\,n_\alpha(\mathbf{k}_1, \mathbf{K})[(\omega_1 - \epsilon_\beta - k_1{}^2/2M + i\gamma_\beta)(\omega_1 - \omega' - \epsilon_\sigma \\ - (\mathbf{k}_1 - \mathbf{k}')^2/2M + i\gamma_\sigma) \times (\omega_1 - \omega' - \omega'' - \epsilon_\lambda \qquad (A.13) \\ - (\mathbf{k}_1 - \mathbf{k}' - \mathbf{k}'')^2/2M + i\gamma_\lambda)(\omega_1 - \omega - \epsilon_\alpha \\ - (\mathbf{k}_1 - \mathbf{k})^2/2M - i\gamma_\alpha)]^{-1} \end{aligned}$$

We can immediately perform the ω_1 integration to get

$$-i(2\pi)^{-3}\int d^3\mathbf{k}_1\, n_\alpha(\mathbf{k}_1, \mathbf{K})[(\omega - \omega_{\beta\alpha} - \mathbf{k}_1\cdot\mathbf{k}/M + i\gamma_{\alpha\beta}) \times (\omega - \omega' - \omega_{\sigma\alpha} - \mathbf{k}_1\cdot(\mathbf{k} - \mathbf{k}')/M + i\gamma_{\sigma\alpha})(\omega''' - \omega_{\lambda\alpha} - \mathbf{k}_1\cdot\mathbf{k}'''/M + i\gamma_{\lambda\alpha})]^{-1} \tag{A.13'}$$

where $\gamma_{\alpha\beta} \equiv \gamma_\alpha + \gamma_\beta$ and we have dropped terms like $k'^2/2M$ as very small compared to $\mathbf{k}_1\cdot\mathbf{k}'/M$.

We are only interested in b^r for frequencies $\omega, \omega', \omega'', \omega'''$ near the resonant laser frequencies. Furthermore (A.13′) is large only for $\omega \approx \omega_{\beta\alpha}$, $\omega - \omega' \approx \omega_{\sigma\alpha}$, $\omega''' \approx \omega_{\lambda\alpha}$. In any physical case the laser operates at only a few closely spaced frequencies in resonance with a single transition (or several related ones). Harmonics of these frequencies will not be in resonance with other transitions. Then the only terms in the sum over states $\alpha, \beta, \sigma, \lambda$ in (A.13′) which will give a large contribution are those where the quantities $\omega_{\beta\alpha}$, $\omega_{\lambda\alpha}$, $\omega_{\sigma\alpha}$ correspond to the resonant transition frequency ω_{21} or $-\omega_{21}$ or 0. Then state λ is state β and σ is α. We ignore all other terms.

Then adding the contribution of the remainder of (2.38) to (A.13′) we find, after some algebra

$$\begin{aligned} b^r_{ijkl}(\mathbf{k}\omega, \mathbf{k}'\omega', \mathbf{k}''\omega'', \mathbf{k}'''\omega''') &= \gamma^i_{\alpha\beta}\gamma^j_{\beta\alpha}\gamma^k_{\alpha\beta}\gamma^l_{\beta\alpha}(2\pi)\delta(\omega - \omega' - \omega'' - \omega''') \\ &\times \int d^3\mathbf{K}\delta(\mathbf{k} - \mathbf{k}' - \mathbf{k}'' - \mathbf{k}''' - \mathbf{K})(2\pi)^{-3} \\ &\cdot \int d^3\mathbf{k}_1[n_\alpha(\mathbf{k}_1, \mathbf{K}) - n_\beta(\mathbf{k}_1, \mathbf{K})] \\ &\times (\omega - \omega_{\beta\alpha} - \mathbf{k}_1\cdot\mathbf{k}/M + i\gamma_{\alpha\beta})^{-1}\{(\omega - \omega' - \mathbf{k}_1 \\ &\cdot(\mathbf{k} - \mathbf{k}')/M + 2i\gamma_\alpha)^{-1}[(\omega''' - \omega_{\beta\alpha} - \mathbf{k}_1\cdot\mathbf{k}'''/M + i\gamma_{\alpha\beta})^{-1} \\ &+ (\omega'' + \omega_{\beta\alpha} - \mathbf{k}_1\cdot\mathbf{k}''/M + i\gamma_{\alpha\beta})^{-1}] \\ &+ (\omega - \omega''' - \mathbf{k}_1\cdot(\mathbf{k} - \mathbf{k}''')/M + 2i\gamma_\beta)^{-1} \\ &\times [(\omega' - \omega_{\beta\alpha} - \mathbf{k}_1\cdot\mathbf{k}'/M + i\gamma_{\alpha\beta})^{-1} \\ &+ (\omega'' + \omega_{\beta\alpha} - \mathbf{k}_1\cdot\mathbf{k}''/M + i\gamma_{\alpha\beta})^{-1}]\} \end{aligned} \tag{A.14}$$

Using (2.3) it is now possible to perform the integrals over $\mathbf{k}_1$. The appearance of $\mathbf{k}_1$ in three denominators, however, makes this a complicated procedure, as it would be necessary to break each term into partial fractions, and the resultant expression would not be very useful. We find a more useful result from the following considerations.

To the extent that the level widths γ_α, γ_β, $\gamma_{\alpha\beta}$ are small relative to the Doppler width the Lorentzian factors in (A.14) vary much more rapidly than the Gaussian $\exp(-\beta k_1^2/2M)$ as a function of $\mathbf{k}_1$. If, furthermore, the various oscillation frequencies are close to the line center $\omega_{\beta\alpha}$, there is only a small error made by considering $n_\alpha(\mathbf{k}_1)$ to be constant during the integration. We proceed in this approximation.

We further note that for the purposes of the integration we may replace each $\mathbf{k}'$, $\mathbf{k}''$, $\mathbf{k}'''$ by $\pm\mathbf{k}$ in such a way that the sum is $\mathbf{k}$. Taking all possible such combinations it is now a simple matter to perform the integrals in (A.14) to find

$$\begin{aligned}
&b^r_{ijkl}(\mathbf{k}\omega, \mathbf{k}'\omega', \mathbf{k}''\omega'', \mathbf{k}'''\omega''') \\
&= -2i\alpha\pi^{1/2} \, | c\mathbf{k} |^{-1} \gamma^i_{\alpha\beta}\, \gamma^j_{\beta\alpha}\, \gamma^k_{\alpha\beta}\, \gamma^l_{\beta\alpha}\, 2\pi\delta(\omega - \omega' - \omega'' - \omega''') \\
&\times \int d^3\mathbf{K}\delta(\mathbf{k} - \mathbf{k}' - \mathbf{k}'' - \mathbf{k}''' - \mathbf{K})[n_\alpha(\mathbf{K}) - n_\beta(\mathbf{K})] \\
&\qquad\qquad \cdot\exp[-\alpha^2(\omega - \omega_{\beta\alpha})^2/c^2k^2] \qquad\qquad (\text{A.15}) \\
&\times \{(\omega + \omega'' + 2i\gamma_{\alpha\beta})^{-1}[(\omega - \omega' + 2i\gamma_\alpha)^{-1} \\
&+ (\omega - \omega''' + 2i\gamma_\beta)^{-1}]_{\mathbf{k}'\approx\mathbf{k}'''\approx-\mathbf{k}''\approx\mathbf{k}} \\
&+ (\omega + \omega' - 2\omega_{\beta\alpha} + 2i\gamma_{\alpha\beta})^{-1}(\omega - \omega''' + 2i\gamma_\beta)^{-1} \,|_{\mathbf{k}''\approx\mathbf{k}'''\approx-\mathbf{k}'\approx\mathbf{k}} \\
&+ (\omega + \omega''' - 2\omega_{\beta\alpha} + 2i\gamma_{\alpha\beta})^{-1}(\omega - \omega' + 2i\gamma_\alpha)^{-1} \,|_{\mathbf{k}'\approx\mathbf{k}''\approx-\mathbf{k}'''\approx\mathbf{k}}\}
\end{aligned}$$

where the exponential is required to be very close to one. Note that there is only an appreciable contribution for $\omega \approx \omega' \approx -\omega'' \approx \omega''' \approx \omega_{\beta\alpha}$. In the traveling wave case only the term $\mathbf{k}' \approx -\mathbf{k}'' \approx \mathbf{k}''' \approx \mathbf{k}$ comes in.

APPENDIX IV. EVALUATION OF $b^{\zeta\zeta'}$ AND $\bar{b}^{\zeta\zeta'}$

We wish to evaluate

$$\bar{b}^{\zeta\zeta_1}(\omega) \equiv \sum_{\zeta_2\zeta_3} b^{\zeta\zeta_2\zeta_1\zeta_3}(\mathbf{k}_0\omega, \mathbf{k}_0\omega_0, -\mathbf{k}_0, \omega - 2\omega_0, \mathbf{k}_0\omega_0) \qquad (\text{A.16a})$$

$$\begin{aligned}
2b^{\zeta\zeta_1}(\omega) \equiv \sum_{\zeta_2\zeta_3} [&b^{\zeta\zeta_1\zeta_2\zeta_3}(\mathbf{k}_0\omega, \mathbf{k}_0\omega, -\mathbf{k}_0, -\omega_0, \mathbf{k}_0\omega_0) \qquad (\text{A.16b}) \\
&+ b^{\zeta\zeta_2\zeta_3\zeta_1}(\mathbf{k}_0\omega, \mathbf{k}_0\omega_0, -\mathbf{k}_0\omega_0, \mathbf{k}_0\omega)]
\end{aligned}$$

where we have used the definitions (3.10) and the preceding discussion. From the definition of $\mathbf{b}$ (2.34c) and using (2.27) we have, schematically

$$\begin{aligned}
\sum_{\zeta_2\zeta_3} b^{\zeta\zeta_2\zeta_1\zeta_3} = i\zeta\gamma^i_{\alpha\beta}\gamma^j_{\beta\sigma}\gamma^k_{\sigma\lambda}\gamma^l_{\lambda\alpha}&(g_\beta{}^r g_\sigma{}^r + g_\beta{}^r\tilde{G}_\sigma + \tilde{G}_\beta g_\sigma{}^a)^{\zeta\zeta_1} \\
&\times (g_\lambda{}^r g_\alpha{}^r + g_\lambda{}^r\tilde{G}_\alpha + \tilde{G}_\lambda g_\alpha{}^a)^{\zeta_1\zeta} \qquad (\text{A.17})
\end{aligned}$$

where, in this context,

$$g^r \equiv g^r\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \tilde{G} \equiv \begin{pmatrix}g^< & -g^<\\ g^> & -g^>\end{pmatrix} \qquad (\text{A.18})$$

As a matrix in $\zeta\zeta_1$, (A.17) is in the canonical form (1.53). From (A.17) we may immediately write

$$\bar{b}^{>(<)}(\mathbf{x}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3) = i\gamma^i_{\alpha\beta}\gamma^j_{\beta\sigma}\gamma^k_{\sigma\lambda}\gamma^l_{\lambda\alpha}(g_\beta{}^r(\mathbf{x}\mathbf{x}_1)g_\sigma^{>(<)}(\mathbf{x}_1\mathbf{x}_2) + g_\beta^{>(<)}(\mathbf{x}\mathbf{x}_1)g_\sigma{}^a(\mathbf{x}_1\mathbf{x}_2)) \times (g_\lambda{}^r(\mathbf{x}_2\mathbf{x}_3)g_\alpha^{<(>)}(\mathbf{x}_3\mathbf{x}) + g_\lambda^{<(>)}(\mathbf{x}_2\mathbf{x}_3)g_\alpha{}^a(\mathbf{x}_3\mathbf{x})) \tag{A.17$'$}$$

We note the symmetry

$$\bar{b}^{>}(\mathbf{x}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3) = \bar{b}^{<}(\mathbf{x}_2\mathbf{x}_3\mathbf{x}\mathbf{x}_1) \tag{A.18}$$

which allows us to find $\bar{b}^>$ from $\bar{b}^<$. Using (2.1) and (2.2′) we may put (A.17′) in a form similar to (A.12). Repeating the arguments in Appendix III we are led to a form similar to but more complicated than (A.14). As in (A.14) we find that $b^<(\omega, \omega', \omega'', \omega''')$ is small unless $\omega \approx \omega' \approx -\omega'' \approx \omega''' \approx \omega_{\beta\alpha}$. Inserting the required values of frequency and wavenumber from (A.16a) we perform the velocity integral as in (A.15) to find

$$\bar{b}^<(\omega) = 2i\,|\gamma^i_{\alpha\beta}|^4\,\alpha\pi^{1/2}\,|c\mathbf{k}|^{-1}\exp[-\alpha^2(\omega-\omega_{\beta\alpha})^2/c^2k^2] \times [n_\beta(\omega-\omega_0-2i\gamma_\alpha)^{-1}(\omega-\omega_0-2i\gamma_\beta)^{-1} + n_\alpha(\omega-\omega_0+2i\gamma_\alpha)^{-1}(\omega-\omega_0+2i\gamma_\beta)^{-1}] \tag{A.19}$$

As in (A.17) we write

$$\sum_{\zeta_2\zeta_3}[b^{\zeta\zeta_1\zeta_2\zeta_3}(\mathbf{x}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3) + b^{\zeta\zeta_2\zeta_3\zeta_1}(\mathbf{x}\mathbf{x}_2\mathbf{x}_3\mathbf{x}_1)] = i\zeta\gamma^i_{\alpha\beta}\gamma^j_{\beta\sigma}\gamma^k_{\sigma\lambda}\gamma^l_{\lambda\alpha} \times [(g_\beta{}^r + \tilde{G}_\beta)^{\zeta\zeta_1}(g_\sigma{}^r g_\lambda{}^r g_\alpha{}^r + g_\sigma{}^r g_\lambda{}^r\tilde{G}_\alpha + g_\sigma{}^r\tilde{G}_\lambda g_\alpha{}^a + \tilde{G}_\sigma g_\lambda{}^a g_\alpha{}^a)^{\zeta_1\zeta} + (g_\beta{}^r g_\sigma{}^r g_\lambda{}^r + g_\beta{}^r g_\sigma{}^r\tilde{G}_\lambda + g_\beta{}^r\tilde{G}_\sigma g_\lambda{}^a + \tilde{G}_\beta g_\sigma{}^a g_\lambda{}^a)^{\zeta\zeta_1}(g_\alpha{}^r + \tilde{G}_\alpha)^{\zeta_1\zeta}] \tag{A.20}$$

Then

$$2b^{>(<)}(\mathbf{x}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3) = i\gamma^i_{\alpha\beta}\gamma^j_{\beta\sigma}\gamma^k_{\sigma\lambda}\gamma^l_{\lambda\alpha}\{g_\beta^{>(<)}(\mathbf{x}\mathbf{x}_1) \times [g_\sigma{}^r(x_1x_2)g_\lambda{}^r(x_2x_3)g_\alpha^{<(>)}(x_3x) + g_\sigma{}^r(x_1x_2)g_\lambda^{<(>)}(x_2x_3)g_\alpha{}^a(x_3x) + g_\sigma^{<(>)}(x_1x_2)g_\lambda{}^a(x_2x_3)g_\alpha{}^a(x_3x)] + [g_\beta{}^r(xx_2)g_\sigma{}^r(x_2x_3)g_\lambda^{>(<)}(x_3x_1) + g_\beta{}^r(xx_2)g_\sigma^{>(<)}(x_2x_3)g_\lambda{}^a(x_3x_1) + g_\beta{}^{>(<)}(xx_2)g_\sigma{}^a(x_2x_3)g_\lambda{}^a(x_3x_1)]g_\alpha^{<(>)}(x_1x)\} \tag{A.21}$$

Note that

$$2b^>(\mathbf{x}\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3) = 2b^<(\mathbf{x}_1\mathbf{x}\mathbf{x}_2\mathbf{x}_3) \tag{A.22}$$

Proceeding as before we find

$$2b^<(\omega) = 2i\,|\gamma^i_{\alpha\beta}|^4\,\alpha\pi^{1/2}\,|c\mathbf{k}|^{-1}\exp[-\alpha^2(\omega-\omega_{\beta\alpha})^2/c^2k^2] \times (n_\beta - n_\alpha)(2\gamma_\alpha\gamma_\beta)^{-1}4\gamma_\alpha{}^2/[(\omega-\omega_0)^2 + 4\gamma_\alpha{}^2] \tag{A.23}$$

As a final note we remark that the velocity integrals in this appendix as well as those in Appendix III in the traveling wave case can be easily done exactly. The differences from our approximate results are indeed small in the region of interest.

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