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A Database for Interpolation of Poiseuille Flow Rates for High Knudsen Number Lubrication Problems

This paper proposes the use of a Poiseuille flow rate database for rapid calculation of a generalized lubrication equation for high Knudsen number gas films. The database is created by numerical calculations based on the linearized Boltzmann equation. The proposed interpolation method is verified to reduce calculation time to several tenths of that required to perform rigorous calculations with the same accuracy.

1 Introduction

The modified Reynolds equation [1], using a first order slip flow boundary condition [2], is one of the approximation equations for thin gas film lubrication problems and is widely used in designing flying head sliders with spacing of less than 0.1 μm[3–5]. Moreover the approximation equation including first and second order slip flow correction terms was also proposed [4, 6]. But, both have restricted use when applied to ultra-thin gas films, because both assume that the Knudsen number, Kn, is small. Recently the authors derived a generalized lubrication equation based on the Boltzmann equation [7, 8]. The Boltzmann equation, which was first introduced to this field by Gans [9], describes the behavior of gas molecules statistically and is valid for arbitrary Knudsen numbers [10]. Thus the generalized lubrication equation clarifies the applicability of approximation equations to ultra-thin spacing lubrication.

The flying characteristics of magnetic head sliders with thin spacing, however, have in the past been analyzed through the approximation equations. The first reason of this is that the first order approximation equation, with an accommodation coefficient $\alpha = 0.89$ [5], and the second order approximation equation, with $\alpha = 1$ [6], have been found to be fairly good approximations for conventional slider design [11]. The other reason is that rigorous calculations require more time than that required for the approximation equations [12]. To design ultra-small and ultra-thin spaced head sliders, which have a promising future, the approximation equations are not applicable and rigorous analysis based on the Boltzman equation is necessary. Therefore reduction of calculation times is one of the most important prerequisites.

In this report, an interpolation method, using Poiseuille flow rates calculated rigorously in advance, is proposed as a method for rapidly calculating flying characteristics with ultra-thin spacing. This method can reduce the calculation time for rigorous analysis to that required for approximation equations.

2 Poiseuille Flow Rate Coefficients

The generalized lubrication equation for ultra-thin spacing is expressed as a mass flow conservation law for the summation of a pressure flow (Poiseuille flow) and a shear flow (Couette flow). For slider bearings with finite widths, shown in Fig. 1, the equation is as follows [12]:

$$\left(\frac{b}{l}\right)^2 \cdot \frac{\partial}{\partial X}\left\{\bar{Q}_p \cdot PH^3 \frac{\partial P}{\partial X}\right\} + \frac{\partial}{\partial Y}\left\{\bar{Q}_p \cdot PH^3 \frac{\partial P}{\partial Y}\right\} = \Lambda_b \cdot \left(\frac{\partial (PH)}{\partial X}\right) + \sigma_b \cdot \left(\frac{\partial (PH)}{\partial \bar{t}}\right) \quad (1)$$

where

$$\bar{Q}_p = \bar{Q}_p(D, \alpha) \equiv \frac{Q_p(D, \alpha)}{Q_{con}(D)}$$

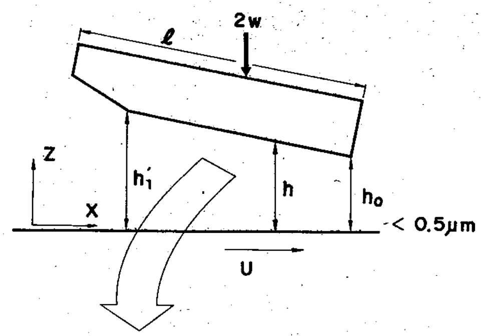

Fig. 1 Coordinates and dimensions employed

Contributed by the Tribology Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS and presented at the STLE/ASME Joint Tribology Conference, Fort Lauderdale, Fla., October 16–19, 1989. Manuscript received by the Tribology Division March 10, 1989. Paper No. 89-Trib-14.

and $Q_p(D, \alpha)$ is a Poiseuille flow rate coefficient and $Q_{con}(=D/6)$ is the coefficient for a continuum flow. Quantities D and α are an inverse Knudsen number and an accommodation coefficient, respectively.

A significant difference between solving equation (1) numerically, and solving conventional lubrication problems, is that the Poiseuille flow rates:

$$Q_p(D, \alpha) \quad \text{and} \quad \bar{Q}_p(D, \alpha)$$

are obtained through the Boltzmann equation. Therefore if the quantities are known, equation (1) can be solved like conventional equations. The inverse Knudsen number, D, in the flow rate coefficients is expressed by using a characteristic inverse Knudsen number, D_0, nondimensional pressure P and spacing H:

$$D = D_0 PH$$

where

$$D_0 = p_{a0} h_{00} / \mu\sqrt{2RT_0}).$$

This quantity is calculated for a reference state of:

$p_{a0} = 101.325\text{KPa}$, $h_{00} = 1\mu\text{m}$, $\mu = 1.81 \times 10^{-5}\text{Pa}\cdot\text{s}$

$T_0 = 293°\text{K}$, $R = 287.03\text{J}/(\text{Kg}\cdot\text{K})$,

resulting in

$$D_0 = 13.657 \cdot p_a \cdot h_0$$

The units for p_a and h_0 are atm. and μm, respectively.

2.1 Equations for Poiseuille Flow Rate Q_p. Poiseuille flow rates were calculated by using an approximation method based on a variational approach [13, 14]. In this paper velocity profiles and the corresponding flow rates for Poiseuille flows are calculated by first solving a linear algebraic equation equivalent to the integro-differential equation governing flow velocities, and then verifying their accuracy. Accurate flow rate values are compiled to create the database.

If the temperatures for a running wall and a slider are uniform in the running direction and their accommodation coefficients are the same, the integro-differential equations for Poiseuille flow velocity profiles are [8]:

$$V_{xp} = (k_0 \epsilon \beta / 2)(1 - \phi p) \tag{2}$$

$$\phi_p(Z, \alpha) = 1 + \frac{1}{\sqrt{\pi} k_0} \int_0^H K(Z, Z', \alpha) \cdot \phi_p(Z', \alpha) dZ' \tag{3}$$

where $K(Z, Z', \alpha)$ is a symmetrical function defined as

$$\begin{aligned} K(Z, Z', \alpha) = {} & T_{-1}\left(\frac{|Z - Z'|}{k_0}\right) \\ & + (1-\alpha)\left\{S_{-1}\left(\frac{Z+Z'}{k_0}, \frac{2H}{k_0}, \alpha\right) + S_{-1}\left(\frac{2H - Z - Z'}{k_0}, \frac{2H}{k_0}, \alpha\right)\right\} \\ & + (1-\alpha)^2\left\{S_{-1}\left(\frac{2H + Z - Z'}{k_0}, \frac{2H}{k_0}, \alpha\right) + S_{-1}\left(\frac{2H - Z + Z'}{k_0}, \frac{2H}{k_0}, \alpha\right)\right\} \end{aligned} \tag{4}$$

The quantities $T_n(x)$ and $S_n(x, y, \alpha)$ are an Abramowitz function and a special case of a generalized Abramowitz function, S_n, respectively, and are defined as

$$T_n(x) = \int_0^\infty t^n \cdot \exp\left(-t^2 - \frac{x}{t}\right) dt \tag{5}$$

$$S_n(x, y, \alpha) = \int_0^\infty \frac{t^n \cdot \exp(-t^2 - x/t)}{1 - (1-\alpha)^2 \cdot \exp(-y/t)} dt \tag{6}$$

If gas molecules reflect diffusely at boundary surfaces ($a = 1$), then equation (4) is simplified to

$$K(Z, Z', \alpha = 1) = T_{-1}\left(\frac{|Z - Z'|}{k_0}\right) \tag{7}$$

Finally, nondimensional flow rate coefficient, $Q_p(D, \alpha)$, is given by the following relation:

Nomenclature

b = width of lubrication region

D = inverse Knudsen number; $D_0PH = ph/(\mu\sqrt{2RT}) = \sqrt{\pi}/2\text{Kn})$

D_0 = characteristic inverse Knudsen number; $p_a h_0/\mu_0\sqrt{2RT_0})$

H = nondimensional film thickness; h/h_0

h = film thickness

h_0 = minimum (characteristic) film thickness

Kn = Knudsen number; λ/n

k = modified Knudsen number; $(2/\sqrt{\pi})\text{Kn} = 1/D$

l = length of lubrication region

P = nondimensional pressure; p/p_a

p = pressure

p_a = ambient pressure

Q_{con} = flow rate coefficients for continuum Poiseuille flows; $-q_{con}\sqrt{2RT}/[h^2(dp/dx)]$

Q_p = flow rate coefficients for Poiseuille flows; $-q_p\sqrt{2RT}/[h^2 dp/dx)]$

$\bar{Q}_p$ = ratio of Q_p and Q_{con}; Q_p/Q_{con}

R = gas constant

S_n = generalized Abramowitz function (see equation (6))

T = temperature

T_0 = characteristic temperature

T_n = Abramowitz function (see equation (5))

U = boundary speed (in the X direction)

V_{xp} = nondimensional flow speed of Poiseuille flow in the X direction; $u_p/\sqrt{2RT}$

W = nondimensional load carrying capacity; $w'/(p_a lb)$

w' = load carrying capacity

X, Y, Z = nondimensional coordinates; x/l, y/b, and z/h_0

x, y, z = coordinates

α = accommodation coefficient at boundary surface ($\alpha = 1$ for diffuse refection, $0 < \alpha < 1$ in general)

β = nondimensional pressure gradient; dP/dX

ϵ = h_0/l

Λ = bearing number; $6\mu Ul/p_a h_0^2$

Λ_b = modified bearing number; $\Lambda \cdot (b/l)^2$

λ = molecular mean free path; $(\sqrt{\pi}/2)\mu\sqrt{2RT_0}/p$

μ = viscosity

σ = squeeze number; $12\mu\omega_0 l^2/p_a h_0^2$

σ_b = modified squeeze number; $\sigma \cdot (b/l)^2$

$\bar{\tau}$ = nondimensional time

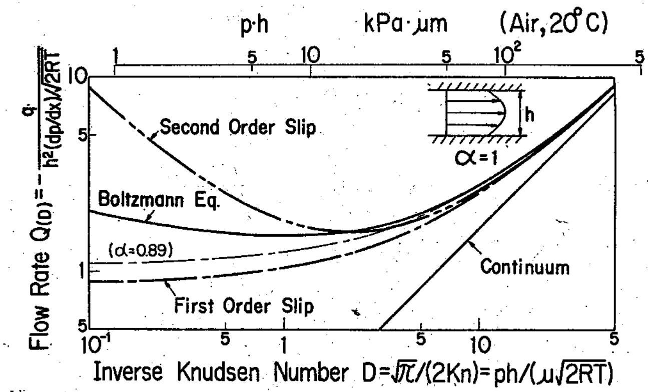

Fig. 2 Poiseuille flow rate coefficients versus inverse Knudsen numbers

$$Q_p(D, \alpha) = -\left(\frac{2}{k_0\epsilon\beta}\right)\frac{1}{D^2}\int_0^H V_{xp}dZ$$

$$= -\frac{1}{D} + \frac{1}{D^2}\left\{\int_0^H \phi_p dZ\right\} \quad (8)$$

2.2 Numerical Method for Poiseuille Flow Rate Q_p. Solving linear algebraic equations equivalent to equation (3), with velocities at each mesh point, $(\psi_p)_j$, as unknown variables [15, 16], is more convenient than employing an iteration method [17]. When dividing the spacing, H, equivalent to the inverse Knudsen number, D, by n meshes, the mesh interval Δ and discrete coordinates z_i and z_j are expressed as

$$\Delta = H/n, \quad z_i = (i-1/2)\cdot\Delta, \quad z_j = (j-1/2)\cdot\Delta \quad (9)$$

then the linear algebraic equation is as follows:

$$\sum_{j=1}^{n} A_{ij}\cdot(\phi_p)_j = 1 \quad (10)$$

where

$$A_{ij} = Q_{ij} - \frac{1}{\sqrt{\pi}}\Bigg[T_0\left(\left|\frac{z_i}{k_0} - \frac{z_j}{k_0}\right| - \frac{\Delta}{2k_0}\right) - T_0\left(\left|\frac{z_i}{k_0} - \frac{z_j}{k_0}\right| + \frac{\Delta}{2k_0}\right)$$
$$+ (1-\alpha)\left\{S_{-1}\left(\frac{Z+Z'}{k_0}, \frac{2H}{k_0}, \alpha\right) + S_{-1}\left(\frac{2H-Z-Z'}{k_0}, \frac{2H}{k_0}, \alpha\right)\right\}\cdot\Delta$$
$$+ (1-\alpha)^2\left\{S_{-1}\left(\frac{2H+Z-Z'}{k_0}, \frac{2H}{k_0}, \alpha\right) + S_{-1}\left(\frac{2H-Z+Z'}{k_0}, \frac{2H}{k_0}, \alpha\right)\right\}\cdot\Delta\Bigg] \quad (11)$$

In particular, for $i=j$

$$T_0\left(\frac{\Delta}{2k_0}\right) + T_0\left(-\frac{\Delta}{2k_0}\right) \doteqdot 0$$

and the first line of equation (11) is simplified as follows

$$A_{ii} = -\frac{1}{\sqrt{\pi}}\left[-2T_0\left(\frac{\Delta}{2k_0}\right) + (1-\alpha)\ldots\ldots + (1-\alpha)^2\ldots\right] \quad (12)$$

For accommodation coefficient $\alpha=1$, equation (11) is simplified to [13, 14]

$$A_{ij} = \delta_{ij} - \frac{1}{\sqrt{\pi}}\left[T_0\left(\left|\frac{z_i}{k_0} - \frac{z_j}{k_0}\right| - \frac{\Delta}{2k_0}\right) - T_0\left(\left|\frac{z_i}{k_0} - \frac{z_j}{k_0}\right| + \frac{\Delta}{2k_0}\right)\right] \quad (13)$$

Moreover, for $i=j$, A_{ii} is as follows:

$$A_{ii} = \frac{2}{\sqrt{\pi}}T_0\left(\frac{\Delta}{2k_0}\right) \quad (14)$$

The velocity profiles, ψ_p, are obtained by solving (10).

2.3 Accurate Numerical Method. In calculating the Poiseuille flow rates, a large number of numerical integrals must be executed. Therefore, numerical errors may occur included in their calculation and the calculation method should be scrutinized. The numerical methods employed in this report are shown in Table 1.

2.4 A Database of Flow Rate Coefficients. Poiseuille

Table 1 Numerical calculation method resulting in high accuracy

For $\alpha = 1$

	D≪1		D		1≪D
Abramowitz function, T_n	Power series approximation		Numerical integral		—
Flow rate coeff. Q_p	Numerical integral	Variation approx.	Numerical integral	Variation approx.	Slip flow approx.
Velocity profiles	O	X	O	X	X

For arbitrary accommodation coefficient

		D≪1	D	1≪D
Abramowitz function	T_n	Power series approximation	Numerical integral	
	S_n	Numerical integral		
Flow rate coeff. Q_p		Numerical integral	Variational approach	
Velocity profile		O	X	

Table 2 A Poiseuille flow rate database for $\alpha = 1$

Inverse Knudsen No D	$\alpha=1$ Flow rate database Q_p		Inverse Knudsen No D	$\alpha=1$ Flow rate database Q_p	
100.0	17.693	A	0.9	1.542	N
90.0	16.028	A	0.8	1.548	N
80.0	14.363	A	0.7	1.559	N
70.0	12.698	A	0.6	1.576	N
60.0	11.033	A	0.5	1.602	N
50.0	9.370	A	0.4	1.641	N
40.0	7.708	A	0.35	1.668	N
35.0	6.878	A	0.3	1.703	N
30.0	6.049	A	0.25	1.748	N
25.0	5.222	A	0.2	1.808	N
20.0	4.398	A	0.15	1.895	N
15.0	3.578	A	0.1	2.033	N
10.0	2.768	A			N
9.0	2.608	A	0.09	2.071	N
8.0	2.449	A	0.08	2.115	N
7.0	2.292	A	0.07	2.167	N
6.0	2.134	A	0.06	2.228	N
	-		0.05	2.302	N
5.0	1.991	N	0.04	2.397	N
4.0	1.846	N	0.035	2.454	N
3.5	1.777	N	0.03	2.522	N
3.0	1.711	N	0.025	2.604	N
2.5	1.649	N	0.02	2.707	N
2.0	1.595	N	0.015	2.846	N
1.5	1.554	N	0.01	3.060	N
1.0	1.539	N			

A : Asymptotic analysis
N : Numerical solutions

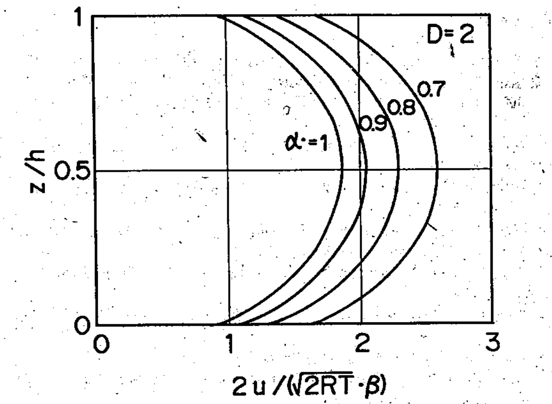

Fig. 3 Velocity profiles for typical accommodation coefficients

flow rate coefficients are calculated numerically for typical inverse Knudsen numbers, D, and accommodation coefficients, α.

(i) For $\alpha = 1$

This corresponds to the case where gas molecules reflect diffusely at surfaces and therefore at almost all engineering surfaces [21]. The numerical results for $\alpha = 1$ are compared with those in references [13, 15, and 16] (see Appendix 1). The final flow rate coefficients comprising the database for $\alpha = 1$ are shown in Table 2.

(ii) For arbitrary accommodation coefficients

The velocity profiles for typical accommodation coefficients α are shown in Fig. 3. The corresponding flow rates, Q_p, are calculated by summing the velocity profiles. In Appendix 2 the numerical flow rate coefficients, Q_p, are compared with those from the variational approach. As the accommodation coefficient α decreases, errors occur when using the variational method.

3. Rapid Calculation Method Using the Flow Rate Database

3.1 Local Interpolation Method. The local interpolation method is explained in Fig. 4. The corresponding flow rate coefficient for the inverse Knudsen number D_1 in a calculated mesh point is calculated by interpolating the database. To increase accuracy, some of the nearest values to D_1 are chosen and their corresponding flow rates are employed in the interpolation. This method can be similarly applied to the case of arbitrary accommodation coefficients, where the time required can be reduced more drastically than that for $\alpha = 1$.

The numerical results and time required by the database method are shown in Table 3. The time required was found to be substantially reduced. This method can be applied to any static and dynamic numerical method.

3.2 Power Series Method. Power series expressions for flow rate coefficients, Q_p, are effective for flow rate calculations. First, for small inverse Knudsen numbers, D, where the slip flow assumption is reasonably valid, the asymptotic expression for $Kn << 1$ can be employed. Power series representations for $0.15 \leq D < 5$ and $0.01 \leq D < 0.15$ were calculated using the method of least squares. The results are as follows:

$$Q_p = D/6 + 1.0162 + 1.0653/D - 2.1354/D^2 \qquad (5 \leqq D)$$

$$Q_p = 0.13852D + 1.25087 + 0.15653/D - 0.00969/D^2 \qquad (0.15 \leqq D < 5)$$

$$Q_p = -2.22919D + 2.10673 + 0.01653/D - 0.0000694/D^2 \qquad (0.01 \leqq D < 0.15)$$

Power series approximations and their errors, as they correspond to the rigorous database approximations, are shown in Figs. 5 (*a*)–(*c*). The error E (percent) is defined as

$$E = (X - Y)/Y \times 100$$

where X is the value of the power series and Y is the numerical results. The symbols + and × designate $E > 0$ and $E < 0$ respectively. As shown in Figs. 5 (*a*)–(*c*), the errors are less than ±1 percent.

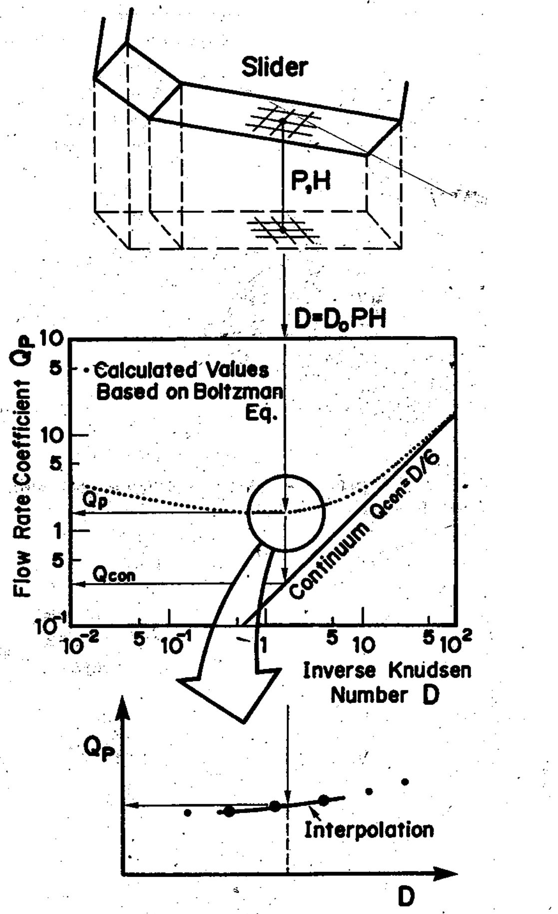

Fig. 4 Rapid calculation method using a Poiseuille flow rate database

Table 3 Calculation results and the time required

(Dimensions employed)

Fixed air film configuration

- Slider length $\ell = 0.5$ mm,
- Taper length $\ell_1 = 0.05$ mm,
- Ski width $b = 51.5\ \mu m$ ($b/\ell = 0.1029$),
- Minimum spacing $h_0 = 0.05\ \mu m$,
- Spacing ratio $h_1'/h_0 = 2.719$,
- Velocity $U = 5$ m/s,

(Number of mesh points)

60 (X coordinate) × 16 (Y coordinate)

(For a half region of each ski)

	$\alpha = 1$			$0 < \alpha < 1$		
	Approx. theory (conven.)	Boltzmann equation: Variation	Boltzmann equation: Database	Approx. (conven.)	Boltzmann equation: Variation	Boltzmann equation: Database
Load capacity, w (gr)	0.04	0.0293	0.0293	0.0338	0.0254	0.0254
Pressure center, $\bar{X}$	0.55	0.5319	0.5319	0.5484	0.5288	0.5288
Calc. time (sec)	19.26	235.5	19.36	19.27	1685.5	19.36
No. of iterations	(3)	(3)	(3)	(3)	(3)	(3)
Calc. time ratio	1	12.2	1.01	1	87.1	1.01
Reference		DT=00.1			DT=0.01	

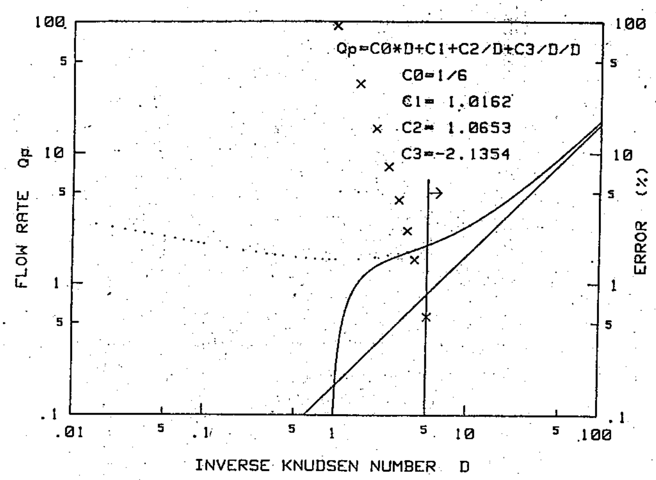

Fig. 5(a) Comparison of power series approximations with numerical calculations ($5 \leq D$)

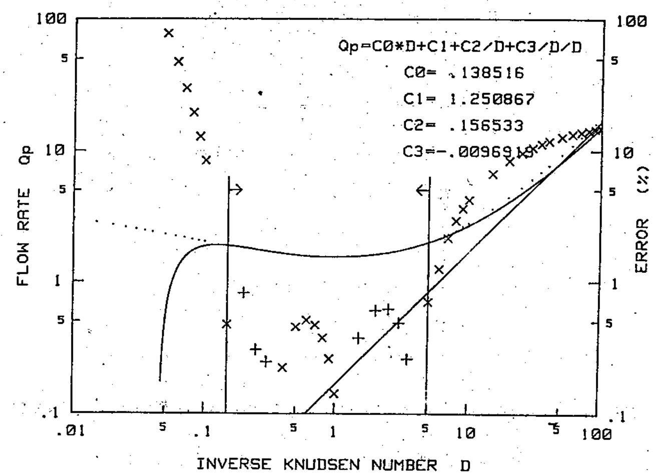

Fig. 5(b) Comparison of power series approximations with numerical calculations ($0.15 \leq D < 5$)

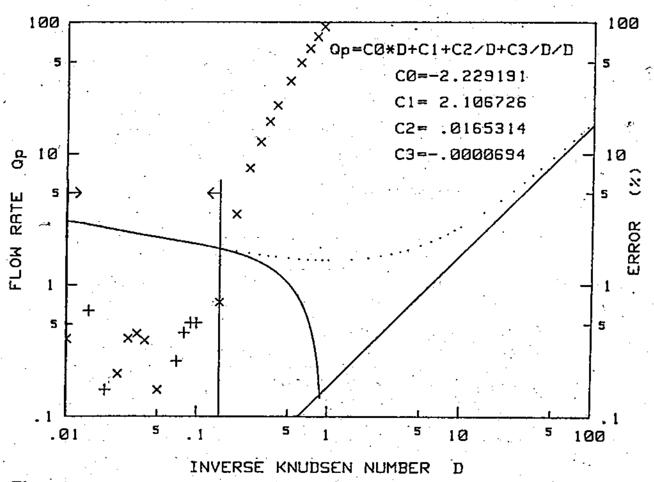

Fig. 5(c) Comparison of power series approximations with numerical calculations ($0.01 \leq D < 0.15$)

Conclusion

In this paper a rapid calculation method for solving ultra-thin lubrication problems by interpolating a flow rate database is proposed. The database is created through numerical calculations based on the linearized Boltzmann equation. The interpolation method reduces the calculation time required, while the accuracy remains high, thereby proving it is a practical design tool for ultra-thin flying head sliders.

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Appendix 1 Table comparing flow rate coefficients Q_p for diffuse reflection ($\alpha = 1$)

Inverse Knudsen No D	$\alpha = 1$				
	Asymptotic (Sone) ⟨X⟩	Variational (Cercignina) ⟨Y⟩	Numerical (Cercignani) ⟨Z⟩	Errors (%) (Z−X)/X	Errors (%) (Y−Z)/Z
100.0	**17.693**		17.071	−3.5	
90.0	**16.028**		15.542	−3.0	
80.0	**14.363**		13.998	−2.5	
70.0	**12.698**		12.438	−2.0	
60.0	**11.033**		10.860	−1.6	
50.0	**9.370**		9.265	−1.1	
40.0	**7.708**	7.700	7.652	−0.7	0.6
35.0	**6.878**	6.870	6.841	−0.5	0.4
30.0	**6.049**	6.042	6.026	−0.4	0.3
25.0	**5.222**	5.215	5.209	−0.3	0.1
20.0	**4.398**	4.391	4.391	−0.1	0.0
15.0	**3.578**	3.572	3.576	−0.1	−0.1
10.0	**2.768**	2.764	2.769	0.0	−0.2
9.0	**2.608**	2.605	2.609	0.0	−0.2
8.0	**2.449**	2.447	2.451	0.1	−0.2
7.0	**2.292**	2.291	2.295	0.2	−0.2
6.0	**2.134**	2.138	2.141	0.3	−0.2
5.0		1.988	**1.991**		−0.1
4.0		1.844	**1.846**		−0.1
3.5		1.775	**1.777**		−0.1
3.0		1.709	**1.711**		−0.1
2.5		1.648	**1.649**		−0.1
2.0		1.594	**1.595**		−0.1
1.5		1.553	**1.554**		0.0
1.0		1.538	**1.539**		0.0
0.9		1.542	**1.542**		0.0
0.8		1.548	**1.548**		0.0
0.7		1.559	**1.559**		0.0
0.6		1.576	**1.576**		0.0
0.5		1.602	**1.602**		0.0
0.4		1.641	**1.641**		0.0
0.35		1.668	**1.668**		0.0
0.3		1.702	**1.703**		0.0
0.25		1.751	**1.748**		0.2
0.2		1.804	**1.808**		−0.2
0.15		1.893	**1.895**		−0.1
0.1		2.032	**2.033**		0.0
0.09		2.071	**2.071**		0.0
0.08		2.115	**2.115**		0.0
0.07		2.167	**2.167**		0.0
0.06		2.229	**2.228**		0.0
0.05		2.301	**2.302**		0.0
0.04		2.397	**2.397**		0.0
0.035		2.455	**2.454**		0.0
0.03		2.524	**2.522**		0.1
0.025		2.607	**2.604**		0.1
0.02		2.711	**2.707**		0.1
0.015		2.849	**2.846**		0.1
0.01		3.050	**3.060**		−0.3

Appendix 2(a) Table comparing flow rate coefficients Q_p for arbitrary accommodation coefficients

Inverse Knudsen No D	$\alpha = 0.9$			$\alpha = 0.8$		
	Numerical (Fukui) (U)	Variational (Fukui) (U')	Errors (%) (U'−U)/U	Numerical (Fukui) (V)	Variational (Fukui) (V')	Errors (%) (V'−V)/V
100.0	**16.779**			**16.906**		
90.0	**15.245**			**15.358**		
80.0	**13.725**			**13.823**		
70.0	**12.263**			**12.416**		
60.0	**10.737**			**10.869**		
50.0	**9.229**			**9.383**		
40.0	**7.659**	7.910	3.3	**7.784**	8.188	5.2
35.0	**6.888**	7.079	2.8	**7.029**	7.357	4.7
30.0	**6.102**	6.250	2.4	**6.251**	6.528	4.4
25.0	**5.304**	5.423	2.2	**5.452**	5.700	4.6
20.0	**4.494**	4.598	2.3	**4.633**	4.875	5.2
15.0	**3.695**	3.778	2.2	**3.845**	4.053	5.4
10.0	**2.898**	2.969	2.4	**3.054**	3.242	6.1
9.0	**2.737**	2.809	2.6	**2.890**	3.072	6.3
8.0	**2.587**	2.651	2.5	**2.751**	2.923	6.3
7.0	**2.431**	2.494	2.6	**2.595**	2.765	6.6
6.0	**2.284**	2.340	2.4	**2.456**	2.610	6.3
5.0	**2.141**	2.189	2.3	**2.320**	2.458	6.0
4.0	**2.007**	2.044	1.9	**2.200**	2.312	5.1
3.5	**1.943**	1.975	1.6	**2.143**	2.241	4.6
3.0	**1.886**	1.909	1.2	**2.096**	2.175	3.7
2.5	**1.834**	1.848	0.7	**2.058**	2.113	2.7
2.0	**1.789**	1.795	0.3	**2.026**	2.059	1.6
1.5	**1.754**	1.756	0.1	**2.001**	2.023	1.1
1.0	**1.746**	1.750	0.2	**2.002**	2.009	0.3
0.9	**1.751**	1.758	0.4	**2.010**	2.020	0.5
0.8	**1.760**	1.780	1.2	**2.021**	2.034	0.6
0.7	**1.774**	1.729	−2.5	**2.039**	2.054	0.7
0.6	**1.795**	1.784	−0.6	**2.065**	2.083	0.9
0.5	**1.826**	1.821	−0.3	**2.102**	2.124	1.0
0.4	**1.873**	1.871	−0.1	**2.158**	2.185	1.2
0.35	**1.906**	1.904	−0.1	**2.197**	2.227	1.4
0.3	**1.947**	1.946	0.0	**2.245**	2.279	1.5
0.25	**2.000**	2.001	0.1	**2.308**	2.347	1.7
0.2	**2.072**	2.074	0.1	**2.393**	2.440	2.0
0.15	**2.175**	2.179	0.2	**2.515**	2.572	2.3
0.1	**2.338**	2.345	0.3	**2.708**	2.783	2.8
0.09	**2.384**	2.391	0.3	**2.762**	2.842	2.9
0.08	**2.436**	2.444	0.3	**2.824**	2.910	3.0
0.07	**2.497**	2.505	0.3	**2.897**	2.989	3.2
0.06	**2.570**	2.578	0.3	**2.984**	3.084	3.4
0.05	**2.659**	2.654	−0.2	**3.090**	3.200	3.6
0.04	**2.772**	2.790	0.7	**3.225**	3.346	2.8
0.035	**2.841**	2.860	0.6	**2.308**	3.435	3.9
0.03	**2.923**	2.942	0.6	**3.406**	3.540	3.9
0.025	**3.021**	3.041	0.7	**3.525**	3.665	4.0
0.02	**3.145**	3.166	0.7	**3.674**	3.827	4.2
0.015	**3.312**	3.333	0.6	**3.874**	4.058	4.7
0.01	**3.564**	3.566	0.0	**4.174**	4.388	5.1

Appendix 2(b) Table comparing flow rate coefficients Q_p for arbitrary accommodation coefficients

Inverse Knudsen No D	$\alpha = 0.7$		
	Numerical (Fukui) (W)	Variational (Fukui) (W')	Errors (%) (W'−W)/W
100.0	**17.051**		
90.0	**15.487**		
80.0	**13.935**		
70.0	**12.597**		
60.0	**11.024**		
50.0	**9.568**		
40.0	**7.933**	8.606	8.5
35.0	**7.200**	7.775	8.0
30.0	**6.432**	6.945	8.0
25.0	**5.632**	6.117	8.6
20.0	**4.803**	5.290	10.1
15.0	**4.029**	4.468	10.9
10.0	**3.246**	3.654	12.6
9.0	**3.078**	3.493	13.5
8.0	**2.951**	3.333	12.9
7.0	**2.794**	3.175	13.6
6.0	**2.666**	3.018	13.2
5.0	**2.540**	2.864	12.8
4.0	**2.437**	2.714	11.4
3.5	**2.390**	2.642	10.6
3.0	**2.357**	2.572	9.1
2.5	**2.338**	2.507	7.3
2.0	**2.324**	2.450	5.4
1.5	**2.314**	2.411	4.2
1.0	**2.330**	2.417	3.7
0.9	**2.340**	2.429	3.8
0.8	**2.355**	2.448	4.0
0.7	**2.376**	2.475	4.2
0.6	**2.407**	2.514	4.4
0.5	**2.452**	2.570	4.8
0.4	**2.518**	2.652	5.3
0.35	**2.563**	2.709	5.7
0.3	**2.620**	2.781	6.1
0.25	**2.695**	2.875	6.7
0.2	**2.795**	3.003	7.4
0.15	**2.939**	3.188	8.5
0.1	**3.168**	3.487	10.1
0.09	**3.232**	3.572	10.5
0.08	**3.306**	3.669	11.0
0.07	**3.393**	3.784	11.5
0.06	**3.497**	3.922	12.2
0.05	**3.624**	4.091	12.9
0.04	**3.786**	4.307	13.8
0.035	**3.886**	4.441	14.3
0.03	**4.004**	4.599	14.9
0.025	**4.148**	4.792	15.5
0.02	**4.329**	5.036	16.3
0.015	**4.571**	5.369	17.4
0.01	**4.932**	5.868	19.0