SUPPLEMENT 2

SOLID
STATE
PHYSICS

GALVANOMAGNETIC EFFECTS
IN SEMICONDUCTORS

SOLID STATE PHYSICS

Advances in Research and Applications

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GALVANOMAGNETIC EFFECTS IN SEMICONDUCTORS

ALBERT C. BEER

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Preface

Galvanomagnetic phenomena in solids have received attention since before the turn of the century, with a significant event being the discovery of the Hall effect in 1879 by E. H. Hall. Subsequent contributions due to Drude and to Lorentz came in the early part of the present century. By the third decade, after significant work of men such as Pauli, Bloch, and Sommerfeld, the basic foundations for electrical transport phenomena in solids were well established, and detailed accounts can be found in such treatises as those by Sommerfeld and Bethe (1933 "Handbuch"), Mott and Jones (1936), Seitz (1940), and Wilson (1953).

The past decade has witnessed greatly expanded activity, mostly in the nature of refinements of the basic theory and application to specific examples, with particular attention to special cases of band structure and charge carrier interactions. The impetus for the accelerated activity has probably been the rapid growth in semiconductor research, an increased availability of good single crystals for measurements, and the development of a number of powerful tools (e.g., the cyclotron resonance) for establishing band structure and scattering mechanisms. As a result of the extensive activity, this monograph contains more than 1000 references published within the last decade.

The purposes in writing this volume are severalfold. The first and foremost objective is to provide a reasonably complete account of recent developments in the field. In this case the book is primarily directed to those readers who are active in solid state physics. It is especially to be hoped that the work will stimulate further research in various areas. A second group of readers are those physicists, chemists, metallurgists, and electronics engineers who may have occasion to interpret the results of galvanomagnetic measurements. For this group the author has tried to emphasize the limitations of the simple formulas, and to point out that unless the experimenter is familiar with the band structure of the material under study, the actual facts may be far from what a naive use of the simple relations for Hall effect or magnetoresistance might suggest. It is for these readers that substantial coverage is given to the effects of complex band structure and unusual scattering mechanisms. Finally, it is felt that the literature available to graduate students in solid state science lacks a treatise which has the completeness necessary for the student to gain adequate advantage of recent developments in transport theory. Therefore, for the volume to be of effective use to students, and to scientists and engineers specializing in other fields, the author has included the development of basic equations, specialized for the simple cases but with apparent applicability for the more complex examples.

In a field where publications are forthcoming at a rapid pace, it is a problem to keep a treatise up to date. The reader will note the large number of 1962 references and the various 1963 references inserted at the proof stage. It was felt that the appropriateness of these published works was such as to justify the inconvenience of the large number of letter suffixes — in one case extending into the second alphabet — and, in the case of the latest insertions, the space-dictated omission of some of the obvious English translations. Because of the volume of pertinent literature, it was not feasible to achieve complete coverage of all the areas encompassed in this monograph. For this reason, the author asks understanding from anyone whose contributions were inadvertently omitted.

In order that the book may be of greatest use to readers encompassing the three categories outlined above, the author has attempted to carry through a continuous and coherent development of the pertinent equations for transport phenomena, ranging from the simple cases of spherical energy surfaces and the relaxation-time approximation to the more complex examples involving various crystal symmetries, many-valley band structures, degenerate bands, anisotropic relaxation times, etc. Included also are discussions of such effects as strong electric- and magnetic-field phenomena, inhomogeneities, complex scattering processes, inapplicability of the relaxation-time concept, literature dealing with non-Boltzmann approaches to transport theory, and numerous other items. Many references are made to thermomagnetic phenomena, since data of these kinds often provide a useful supplement to galvanomagnetic measurements in furthering our understanding of the processes responsible for the observed transport behavior in a semiconductor. In a number of cases it has also been desirable to include information on conduction phenomena in metals, especially where the immediate concern has been scattering, magnetoresistance, or other investigations delineating the Fermi-surface topology.

There are many persons whose assistance and encouragement are responsible for making possible the writing of this volume. Besides those whose scientific contributions are acknowledged at the conclusion of the text, the writer wishes to express his indebtedness to the Battelle Memorial Institute for providing the facilities and the environment which made it possible to pursue the task. Thanks are also due the Air Force Office of Scientific Research for their support of research which has resulted in the contribution of a number of sections to this volume. Special words of praise are warranted for the library staff at Battelle for their patient untiring efforts and for the author's secretary for her careful checking of the proofs. Finally, the author wishes to mention his wife and his uncle, Jesse Beer, whose varied contributions were equally necessary to the achievement of this work.

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October, 1963

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GALVANOMAGNETIC EFFECTS IN SEMICONDUCTORS

I.

Introduction

Studies of galvanomagnetic effects have been very helpful in furthering our knowledge of conduction phenomena in semiconductors. Examples of the use of the Hall effect to determine charge-carrier densities and of transverse magnetoresistance to calculate mobilities are so well known as to require no emphasis here. Advantages occurring from measurements of the magnetic-field dependence of the galvanomagnetic effects, on the other hand, are not so obvious. Yet such investigations can yield a variety of information in the case of high-mobility semiconductors. This follows from the fact that in the weak-field region, conductivity phenomena are strongly influenced by the interactions of charge carriers and the lattice, for example, by the nature of the scattering process and by the dependence of the electron wave number on energy. In the region of strong magnetic fields, on the other hand, certain of the transport processes become indicative of the charge-carrier characteristics per se, such as their density and Fermi energy. The magnetic-field variation of the galvanomagnetic effects is expressed in terms of a dimensionless parameter involving the product of the mobility and the magnetic field strength. As a result the strong-field region can be achieved in high-mobility materials with normal laboratory magnets.

For intrinsic semiconductors, magneto-experiments on a single specimen can, in certain cases, give us information about both the conduction and the valence bands. This is possible when electrons and holes have widely differing mobilities. Examples of such materials are indium antimonide and indium arsenide, where the mobility due to lattice scattering is over 80 times as large for electrons as for holes. At weak magnetic fields, the galvanomagnetic and thermomagnetic characteristics are predominantly representative of the conduction band. As the field is increased, the electronic contributions to transport processes may saturate or decrease monotonically so that, at high fields, the controlling effects are due to the holes.

It is the intent of the author to devote most of the space in this book to contributions coming in the last 15 years. The principal concern will be with considerations occasioned by band structure and scattering mechanisms exemplified by a number of the well-known semiconductors. Nevertheless, the transport equations will be outlined quite generally, to serve as a starting point for more specialized derivations. The basic relationships are of course applicable to metals as well as semiconductors; and for a thorough presentation of this phase, the reader is referred to the article by Jan in Volume V of this series.[1]

Thermomagnetic phenomena — which would require the space of a parallel article — are not considered in detail except in those cases where they cause significant differences for isothermal and for adiabatic conditions. In order that the magnitude of such differences can be assessed for a given semiconductor, the temperature gradient terms will be included in the basic transport equations.

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[1] J. P. Jan, *Solid State Phys.* 5, 1 (1957).

II.
Transport Equations

1. Phenomenological Equations for General Case of Anisotropic Solids

The density of the electric current **J** and that of the heat current **q** may be written in terms of the gradients of the electrochemical potential and the temperature as follows:[2]

$$J_i = \sigma_{ij}(\mathbf{H}) E_j^* + \mathscr{M}_{ij}(\mathbf{H}) \frac{\partial T}{\partial x_j} \tag{1.1}$$

$$i = 1, 2, 3$$

$$q_i = \mathscr{N}_{ij}(\mathbf{H}) E_j^* + \mathscr{L}_{ij}(\mathbf{H}) \frac{\partial T}{\partial x_j} \tag{1.2}$$

where the current and field vectors are represented by their components in a Cartesian coordinate system (x_1, x_2, x_3). The electrochemical fields E_j^* are related to the electric fields E_j and to the electrochemical potential Λ of the electrons as follows:

$$E_j^* = E_j + \frac{1}{e}\frac{\partial \zeta}{\partial x_j} \equiv \frac{1}{e}\frac{\partial}{\partial x_j}(-e\varphi + \zeta) \equiv \frac{1}{e}\frac{\partial \Lambda}{\partial x_j}, \qquad e > 0 \text{ for electrons} \tag{1.3}$$

where $-e$ is the charge on the electron, ζ is the chemical potential or Fermi energy, and φ is the electrostatic potential. The tensor transport coefficients σ, $\mathscr{M}$, $\mathscr{N}$, and $\mathscr{L}$ are functions of the magnetic field vector. Phenomenological relations such as Eqs. (1.1) and (1.2) are readily established by use of the thermodynamics of irreversible processes. For details, the reader may consult, for example, the *Handbuch* article of

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[2] This standard convention implying summation over repeated indices is followed throughout this book.

Meixner and Reik,[2a] or other literature to be cited subsequently in this section.

A certain convenience is afforded by the inclusion of the spatial derivative of the Fermi energy in the force terms, and results are consistent with the interpretation of most standard measurements.[3] The case of a two-band semiconductor where the energy gap is a function of temperature has been discussed by Tauc,[3a] and the concept of an internal electric field was introduced. Such a consideration was useful also in the theory of photovoltaic effects when the band gap varied as a result of such factors as external pressure, as in the so-called photopiezoelectric effect.[3a] The case of strongly interacting particles in a strong magnetic field has been discussed by Kasuya[3b] and Nakajima.[3c] In such situations the electrical currents due to the external electric field and the gradient of the chemical potential are different in general, and the Einstein relation between electrical conductivity and charge-carrier diffusion coefficient is not in general applicable — being in general valid only for the symmetric part of the diffusion and electrical conductivity tensors.[3b,c]

Our heat current density $\mathbf{q}$ (identical with Jan's $\mathbf{w}^*$) is defined in terms of the entropy current density $\mathbf{J}_s$ by the thermodynamic relation[4]

$$\mathbf{q} = T\mathbf{J}_s. \tag{1.4}$$

Assuming transport by charge carriers of a single sign, the quantity $\mathbf{q}$ is related to the heat flow $\mathbf{J}_q$ as used by de Groot,[5] Fieschi,[6] and Domenicali,[7] and the heat current densities $\mathbf{J}_w$ of Kohler[8] and $\mathbf{w}$ of Wilson[9] as follows:

[2a] J. Meixner and H. G. Reik, *in* "Handbuch der Physik" (S. Flügge, ed.), Vol. III/2, p. 413. Springer, Berlin, 1959.

[3] See H. B. Callen, *Phys. Rev.* **85**, 16 (1952).

[3a] J. Tauc, *J. Phys. Soc. Japan* **14**, 1174 (1959).

[3b] T. Kasuya, *J. Phys. Soc. Japan* **14**, 410 (1959).

[3c] S. Nakajima, *Progr. Theoret. Phys.* (*Kyoto*) **20**, 948 (1958).

[4] H. B. Callen, *Phys. Rev.* **73**, 1349 (1948). Callen uses **Q** and **S**, respectively, in his notation.

[5] S. R. de Groot, "Thermodynamics of Irreversible Processes," p. 152. Interscience, New York, 1951.

[6] See p. 12 of R. Fieschi, *Nuovo cimento Suppl.* [10] **1**, 1 (1955).

[7] C. A. Domenicali, *Revs. Modern Phys.* **26**, 237 (1954).

[8] M. Kohler, *Ann. Physik* **40**, 601 (1941).

[9] A. H. Wilson, "The Theory of Metals," 2nd ed. Cambridge Univ. Press, London and New York, 1953.

$$\mathbf{J}_q \equiv \mathbf{J}_w \equiv \mathbf{w} = \mathbf{q} - \frac{1}{e}\zeta\mathbf{J}, \qquad e > 0 \text{ for electrons} \tag{1.5}$$

where $-(1/e)\mathbf{J}$ is the particle flow density $\mathbf{J}_k$ as used by de Groot and Fieschi. Another quantity of interest, especially in connection with thermoelectric phenomena, is the total energy flux density $\mathbf{U}$, given by

$$\mathbf{U} = \mathbf{q} - \frac{\Lambda}{e}\mathbf{J} = \mathbf{w} + \varphi\mathbf{J}. \tag{1.6}$$

It is important to note that in Eqs. (1.1), (1.5), and (1.6) the electric current density $\mathbf{J}$ refers to transport by a single type of charged carrier. Furthermore, if additional types of carriers of different signs are present, the $\mathbf{J}$ in relations (1.5) and (1.6), in contrast to that in relation (1.1), cannot in general be expressed as an algebraic sum of the partial electric currents due to each type of carrier. In such cases, the individual particle currents $\mathbf{J}_i/(-e_i)$ must be summed. The $(-\zeta/e)\mathbf{J}$ term in (1.5) is thus replaced by[10] $\Sigma\zeta_k\,\mathbf{J}_k$ in de Groot's and Fieschi's notations where $\mathbf{J}_k$ is a particle current. The summation is done over the species of particles. As is discussed in the next paragraph, these considerations are important in the case of intrinsic semiconductors.

Laboratory measurements of certain quantities, e.g., thermal conductivity, are made under conditions that $\mathbf{J}$ be zero. In such cases, for *extrinsic* semiconductors (i.e., transport by charge carriers of a single sign), the distinction among the different heat currents discussed above ceases to exist.

When *intrinsic* conduction exists, however, both electrons and holes can drift down the temperature gradient to recombine in the cooler region. In such a process the total electric current vanishes when the individual electron and hole currents are equal and opposite. However, particle transport and energy transport are nonzero. This effect can be quite noticeable in intrinsic semiconductors.[11–15]

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[10] See Eq. II.28 of reference 6.

[11] P. J. Price, *Phys. Rev.* **95**, 596 (1954); *Phil. Mag.* **46**, 1252 (1955).

[12] H. Fröhlich and C. Kittel, *Physica* **20**, 1086 (1954).

[13] J. M. Thuillier, *Compt. rend. acad. sci.* (*Paris*) **241**, 1121 (1955); **242**, 2633 (1956).

[14] A. F. Ioffe, "Semiconductor Thermoelements," translated edition, p. 46. Infosearch Limited, London, 1957.

[15] H. J. Goldsmid, *Proc. Phys. Soc.* (*London*) **B69**, 203 (1956).

Use of the heat current density $\mathbf{q}$ allows us to express relationships (1.1) and (1.2) in a convenient form for application of the thermodynamics of irreversible processes by the choice of grad Λ $(\equiv e\mathbf{E}^*)$ and grad T as forces. Since $[1/(-e)]\mathbf{J}$ and $\mathbf{J}_s[\equiv (1/T)\mathbf{q}]$ are mass and entropy flows, respectively, Onsager's relations yield the following equalities[16,16a] between the transport coefficients in a magnetic field $\mathbf{H}$:

$$\begin{aligned} \sigma_{ij}(\mathbf{H}) &= \sigma_{ji}(-\mathbf{H}), \\ \mathscr{L}_{ij}(\mathbf{H}) &= \mathscr{L}_{ji}(-\mathbf{H}), \\ \mathscr{N}_{ij}(\mathbf{H}) &= -T\mathscr{M}_{ji}(-\mathbf{H}). \end{aligned} \tag{1.7}$$

The relationship between the transport tensors used in Eqs. (1.1) and (1.2) and those of Kohler[8,17] are

$$\begin{aligned} &\boldsymbol{\sigma} \equiv \mathbf{S}^{(1)} \\ &\mathscr{M} \equiv -\frac{1}{T}\left[\mathbf{S}^{(2)} + \frac{\zeta}{e}\mathbf{S}^{(1)}\right] \\ &\mathscr{N} \equiv \mathbf{S}^{(3)} + \frac{\zeta}{e}\mathbf{S}^{(1)} \\ &\mathscr{L} \equiv -\frac{1}{T}\left[\mathbf{S}^{(4)} + \frac{\zeta}{e}(\mathbf{S}^{(3)} + \mathbf{S}^{(2)}) + \left(\frac{\zeta}{e}\right)^2 \mathbf{S}^{(1)}\right]. \qquad e > 0 \text{ for electrons} \end{aligned} \tag{1.8}$$

By applying the Kohler symmetry relations, namely,

$$\begin{aligned} S_{kl}^{(m)}(\mathbf{H}) &= S_{lk}^{(m)}(-\mathbf{H}) \qquad m = 1, 4 \\ S_{kl}^{(2)}(\mathbf{H}) &= S_{lk}^{(3)}(-\mathbf{H}) \end{aligned}$$

one also obtains Eqs. (1.7).

---

[16] Reference 6, p. 29. Note that Fieschi's J_e is a particle flow. See also, R. Fieschi, S. R. de Groot, and P. Mazur, *Physica* **20**, 67 (1954); S. R. de Groot and P. Mazur, *Phys. Rev.* **94**, 218 (1954).

[16a] For a more general theory of irreversible processes of which Onsager's relations are a special case, consult R. Zwanzig, *Phys. Rev.* **124**, 983 (1961).

[17] Similar results are given by Wilson[8] (p. 194) except that negative signs were apparently inadvertently omitted from the terms involving the tensors $\mathbf{S}^{(2)}$ and $\mathbf{S}^{(4)}$ in Eqs. (8.1.9) and (8.1.10).

An obvious consequence of the Kohler-Onsager relations is that in the absence of a magnetic field the conductivity tensor $\boldsymbol{\sigma}$ and the thermal flow tensor $\mathscr{L}$ be symmetric. When a field is present, it follows that all the diagonal matrix components of these tensors are even functions of the field; that is,

$$\sigma_{ll}(\mathbf{H}) = \sigma_{ll}(-\mathbf{H}), \tag{1.9}$$

$$\mathscr{L}_{ll}(\mathbf{H}) = \mathscr{L}_{ll}(-\mathbf{H}).$$

Corresponding statements about the cross-effect tensors $\mathscr{M}$ and $\mathscr{N}$, which are related to the thermoelectric processes, cannot apparently be made without placing requirements on the symmetry of the crystal.[18–21]

2. Inversion of Fundamental Transport Equations

The basic transport relationships in the previous section present the currents in terms of electric and thermal gradients. This approach is very common in the literature. From a formalistic point of view, it has advantages in that the transport tensors in such a representation are more simply expressed in terms of material parameters such as Fermi energies, charge-carrier densities and mobilities, and band structure considerations. On the other hand, most measurements are carried out under conditions such that the electric current is the independent variable. For this reason, there is a practical advantage in inverting the basic transport relations so that the electrochemical potential gradients are no longer independent variables. This is done by multiplying Eq. (1.1) by components of the resistivity tensor ρ_{li} and summing over i. Since $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ are reciprocal, one obtains

$$\rho_{li} J_i = \delta_{lj} E_j^* + \rho_{li} \mathscr{M}_{ik} (\partial T/\partial x_k), \qquad \begin{array}{l} \delta_{ij} = 1 \text{ for } i = j \\ \delta_{ij} = 0 \text{ for } i \neq j \end{array}$$

or

$$E_j^* = \rho_{jk} J_k - \rho_{ji} \mathscr{M}_{ik} (\partial T/\partial x_k). \tag{2.1}$$

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[18] E. A. Uehling, *Phys. Rev.* **39**, 821 (1932).

[19] M. Kohler, *Ann. Physik* **27**, 201 (1936).

[20] J. Meixner, *Ann. Physik* **40**, 165 (1941).

[21] See also comments on p. 14 of reference 1. Note, however, reference 432a.

Hence

$$q_i = \mathscr{N}_{ij}\,\rho_{jk}\,J_k - (\mathscr{N}_{ij}\,\rho_{jl}\,\mathscr{M}_{lk} - \mathscr{L}_{ik})\,(\partial T/\partial x_k);$$

that is,

$$E_i^* = \rho_{ik}\,J_k - \rho_{ij}\,\mathscr{M}_{jk}\,(\partial T/\partial x_k) \tag{2.2}$$

$$q_i = \mathscr{N}_{ij}\,\rho_{jk}\,J_k - (\mathscr{N}_{ij}\,\rho_{jl}\,\mathscr{M}_{lk} - \mathscr{L}_{ik})\,(\partial T/\partial x_k). \tag{2.3}$$

We choose to write these relations in terms of tensors of resistivity $\boldsymbol{\rho}$, absolute thermoelectric power $\boldsymbol{\alpha}$, absolute Peltier coefficient $\boldsymbol{\pi}$, and thermal conductivity $\boldsymbol{\kappa}$, as follows:

$$E_i^* = \rho_{ik}(\mathbf{H})J_k + \alpha_{ik}(\mathbf{H})\,\frac{\partial T}{\partial x_k} \tag{2.4}$$

$$i = 1, 2, 3.$$

$$q_i = \pi_{ik}(\mathbf{H})J_k - \kappa_{ik}(\mathbf{H})\,\frac{\partial T}{\partial x_k} \tag{2.5}$$

The relationship between the coefficients in the two representations is as follows:

$$\rho_{ij}\,\sigma_{jk} = \delta_{ik}, \tag{2.6}$$

$$\alpha_{ik} = -\,\rho_{ij}\,\mathscr{M}_{jk}, \tag{2.7}$$

$$\pi_{ik} = \mathscr{N}_{ij}\,\rho_{jk}, \tag{2.8}$$

$$\kappa_{ik} = \mathscr{N}_{ij}\,\rho_{jl}\,\mathscr{M}_{lk} - \mathscr{L}_{ik}. \tag{2.9}$$

Application of the Kohler-Onsager relations (1.7) yields the following reciprocal relations:

$$\rho_{ik}(\mathbf{H}) = \rho_{ki}(-\,\mathbf{H}), \tag{2.10}$$

$$\kappa_{ik}(\mathbf{H}) = \kappa_{ki}(-\,\mathbf{H}), \tag{2.11}$$

$$\pi_{ik}(\mathbf{H}) = T\alpha_{ki}(-\,\mathbf{H}). \tag{2.12}$$

The last equation is the form of the Kelvin second relation for applied magnetic fields.[21a]

[21a] The designation Kelvin *second* relation follows the practice employed by de Groot, Fieschi, and others. Some authors, however, refer to the expression as the *first* relation.

The fundamental transport equations (1.1) and (1.2) are sometimes expressed in terms of the measureable quantities α, π, and κ, namely,

$$J_i = \sigma_{ij}(\mathbf{H})E_j{}^* - \sigma_{il}(\mathbf{H})\,\alpha_{lj}(\mathbf{H})\,\frac{\partial T}{\partial x_j}, \tag{2.13}$$

$$q_i = \pi_{il}(\mathbf{H})\sigma_{lj}(\mathbf{H})E_j{}^* - [\pi_{il}(\mathbf{H})\sigma_{lm}(\mathbf{H})\,\alpha_{mj}(\mathbf{H}) + \kappa_{ij}(\mathbf{H})]\,\frac{\partial T}{\partial x_j}. \tag{2.14}$$

The sign convention adopted in Eqs. (2.4) and (2.5) is the same as that used by Kohler,[8] Wilson,[22] Price,[23] Samoilowitsch and Korenblit,[24] and others. The absolute thermoelectric power is positive for transport by holes, negative for transport by electrons. The difference in absolute Peltier coefficients for metals A and B, namely, $\pi_A - \pi_B$,[25] is defined as the heat *emitted* per second at the junction when a unit current is directed from conductor A to conductor B. The π_{AB} used by Callen,[4] by Fieschi,[26] and by de Groot[5] is defined as the heat *absorbed* when current passes from conductor A to conductor B. However, this relative coefficient is equivalent to $\pi_B - \pi_A$ and, therefore, the sign convention is identical with ours. A similar convention is used in Blatt's article.[27] The relative thermoelectric power used by de Groot, namely, $\Delta\varphi/\Delta T$, is apparently defined in terms of the absolute quantities as $\alpha_A - \alpha_B$, leading to the negative sign in some of his relations as applied to couples.

On the other hand, the sign convention for Domenicali's absolute Peltier coefficient is opposite from ours,[7] inasmuch as he defines $\pi_{XR} \equiv \pi_X^{\text{abs}} - \pi_R^{\text{abs}}$. This leads to a negative sign in his expression for the Kelvin second relation, since his thermoelectric power S^{abs} is defined similarly to our α.

The absolute thermoelectric coefficients are simply related to the entropy of transfer or transport, S^*, which is the entropy carried by a unit flow of particles at the isothermal state. The relation is

$$\pi_{ik} = -(T/e)S_{ik}{}^*, \qquad e > 0 \text{ for electrons.} \tag{2.15}$$

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[22] Reference 9, p. 202.

[23] P. J. Price, *Phys. Rev.* **104**, 1223 (1956).

[24] A. G. Samoilowitsch and L. L. Korenblit, *Fortschr. Physik* **1**, 487 (1954).

[25] Written as π_{AB} in Wilson's notation; as $\pi_B{}^A$ in Price's.

[26] Reference 6, p. 45.

[27] F. J. Blatt, *Solid State Phys.* **4**, 228 (1959).

Our thermoelectric tensor α_{ik} is equivalent to $-1/e\, \mathfrak{S}_{ik}$ of Wilson,[9] and to the negative of Meixner's ε_{ik}.[20, 28, 29] It is also seen that our π is the negative transpose of Meixner's π.[20, 28] This definition is consistent with his expression of the Kelvin second relation as $T\varepsilon_{ik}(\mathbf{H}) = \pi_{ik}(-\mathbf{H})$.[20]

3. Application to Isotropic Solids

Consider an isotropic medium[30] with the magnetic field vector along the z-coordinate direction. Isotropy then requires that

$$\sigma_{xx}(\mathbf{H}) = \sigma_{yy}(\mathbf{H}), \qquad \sigma_{xy}(\mathbf{H}) = -\sigma_{yx}(\mathbf{H}) \tag{3.1}$$

$$\mathbf{H} \equiv (0, 0, H_z)$$

$$\sigma_{xz}(\mathbf{H}) = \sigma_{zx}(\mathbf{H}) = \sigma_{yz}(\mathbf{H}) = \sigma_{zy}(\mathbf{H}) = 0 \tag{3.2}$$

with similar expressions for $\mathscr{M}$, $\mathscr{N}$, and $\mathscr{L}$.

The $\sigma_{zz}(\mathbf{H})$ will, in general, be different from the other two diagonal elements of the conductivity tensor except in the trivial case when $H_z = 0$. In the latter case, all off-diagonal elements will of course vanish in this isotropic example.

The symmetry considerations also require that the diagonal elements of the transport tensors be even functions of magnetic field; the off-diagonal elements, odd. Thus

$$\sigma_{ii}(\mathbf{H}) = \sigma_{ii}(-\mathbf{H}), \qquad \sigma_{ij}(\mathbf{H}) = -\sigma_{ij}(-\mathbf{H}) \qquad \mathbf{H} = (0, 0, H_z)$$

$$i = x, y \tag{3.3}$$

$$\sigma_{zz}(\mathbf{H}) = \sigma_{zz}(-\mathbf{H}) \qquad\qquad j = x, y.$$

with similar expressions for $\mathscr{M}$, $\mathscr{N}$, and $\mathscr{L}$.

The Kohler-Onsager relations, Eqs. (1.7) in conjunction with (3.1) and (3.2), also yield the same relations. In addition, they supply the further condition

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[28] J. Meixner, *Ann. Physik* **35**, 701 (1939).

[29] See reference 1. Jan also defines an absolute thermoelectric power S as the negative of Meixner's (see p. 62 of reference 1).

[30] Actually, less restrictive conditions are sufficient if certain crystallographic symmetry exists. See comment by H. B. Callen[4] (p. 1356).

$$\begin{aligned} \mathscr{N}_{ij}(\mathbf{H}) &= -T\mathscr{M}_{ij}(\mathbf{H}) \qquad && \mathbf{H} = (0, 0, H_z) \\ & && i = x, y \qquad\qquad (3.4) \\ \mathscr{N}_{zz}(\mathbf{H}) &= -T\mathscr{M}_{zz}(\mathbf{H}) \qquad && j = x, y. \end{aligned}$$

Thus the use of the Kohler-Onsager relations and the isotropy of the medium has reduced the large number of coefficients in Eqs. (1.1) and (1.2) to nine independent quantities, as seen below:

$$J_x = \sigma_{xx}(\mathbf{H})E_x^* + \sigma_{xy}(\mathbf{H})E_y^* + \mathscr{M}_{xx}(\mathbf{H})\frac{\partial T}{\partial x} + \mathscr{M}_{xy}(\mathbf{H})\frac{\partial T}{\partial y}$$

$$\mathbf{H} = (0, 0, H_z)$$

$$J_y = -\sigma_{xy}(\mathbf{H})E_x^* + \sigma_{xx}(\mathbf{H})E_y^* - \mathscr{M}_{xy}(\mathbf{H})\frac{\partial T}{\partial x} + \mathscr{M}_{xx}(\mathbf{H})\frac{\partial T}{\partial y}$$

$$q_x = -T\mathscr{M}_{xx}(\mathbf{H})E_x^* - T\mathscr{M}_{xy}(\mathbf{H})E_y^* + \mathscr{L}_{xx}(\mathbf{H})\frac{\partial T}{\partial x} + \mathscr{L}_{xy}(\mathbf{H})\frac{\partial T}{\partial y}$$

$$q_y = T\mathscr{M}_{xy}(\mathbf{H})E_x^* - T\mathscr{M}_{xx}(\mathbf{H})E_y^* - \mathscr{L}_{xy}(\mathbf{H})\frac{\partial T}{\partial x} + \mathscr{L}_{xx}(\mathbf{H})\frac{\partial T}{\partial y} \qquad (3.5)$$

and for the longitudinal effects, we have

$$J_z = \sigma_{zz}(\mathbf{H})E_z^* + \mathscr{M}_{zz}(\mathbf{H})\frac{\partial T}{\partial z}$$

$$\mathbf{H} = (0, 0, H_z).$$

$$q_z = -T\mathscr{M}_{zz}(\mathbf{H})E_z^* + \mathscr{L}_{zz}(\mathbf{H})\frac{\partial T}{\partial z} \qquad (3.6)$$

The inverted form, Eqs. (2.4) and (2.5), for the isotropic case is as follows:

$$E_x^* = \rho_{xx}(\mathbf{H})J_x + \rho_{xy}(\mathbf{H})J_y + \alpha_{xx}(\mathbf{H})\frac{\partial T}{\partial x} + \alpha_{xy}(\mathbf{H})\frac{\partial T}{\partial y}$$

$$\mathbf{H} = (0, 0, H_z)$$

$$E_y^* = -\rho_{xy}(\mathbf{H})J_x + \rho_{xx}(\mathbf{H})J_y - \alpha_{xy}(\mathbf{H})\frac{\partial T}{\partial x} + \alpha_{xx}(\mathbf{H})\frac{\partial T}{\partial y}$$

$$q_x = \pi_{xx}(\mathbf{H}) J_x + \pi_{xy}(\mathbf{H}) J_y - \kappa_{xx}(\mathbf{H}) \frac{\partial T}{\partial x} - \kappa_{xy}(\mathbf{H}) \frac{\partial T}{\partial y} \tag{3.7}$$

$$q_y = -\pi_{xy}(\mathbf{H}) J_x + \pi_{xx}(\mathbf{H}) J_y + \kappa_{xy}(\mathbf{H}) \frac{\partial T}{\partial x} - \kappa_{xx}(\mathbf{H}) \frac{\partial T}{\partial y}$$

$$E_z^* = \rho_{zz}(\mathbf{H}) J_z + \alpha_{zz}(\mathbf{H}) \frac{\partial T}{\partial z}$$

$$q_z = \pi_{zz}(\mathbf{H}) J_z - \kappa_{zz}(\mathbf{H}) \frac{\partial T}{\partial z}$$

where

$$\begin{aligned} \pi_{ij}(\mathbf{H}) &= T\alpha_{ij}(\mathbf{H}) \qquad && \mathbf{H} = (0, 0, H_z) \\ & && i = x, y \\ \pi_{zz}(\mathbf{H}) &= T\alpha_{zz}(\mathbf{H}) \qquad && j = x, y. \end{aligned} \tag{3.8}$$

In this example, with the transverse magnetic field, the existence of relations (3.2) yields a simple relation between the conductivity and resistivity tensors. In particular, relation (2.6) reduces to

$$\sigma_{xx} = \frac{\rho_{yy}}{\rho_{xx}\rho_{yy} - \rho_{xy}\rho_{yx}}, \qquad \sigma_{xy} = \frac{-\rho_{xy}}{\rho_{xx}\rho_{yy} - \rho_{xy}\rho_{yx}} \qquad \mathbf{H} \equiv (0, 0, H_z). \tag{3.9}$$

Due to isotropy in the xy-plane, i.e., applicability of relations (3.1), the following well-known expressions are obtained:

$$\sigma_{xx} = \frac{\rho_{xx}}{\rho_{xx}^2 + \rho_{xy}^2}, \qquad \sigma_{xy} = \frac{-\rho_{xy}}{\rho_{xx}^2 + \rho_{xy}^2}, \qquad \mathbf{H} = (0, 0, H_z). \tag{3.10}$$

4. Physical Significance of Transport Coefficients

The purpose of this section is to discuss the physical interpretation of the transport coefficients in the fundamental equations when simple boundary conditions are applied.

a. One-Dimensional Case, Absence of Magnetic Field

Transport equations (3.7) reduce to

$$E_x^* = \rho_{xx} J_x + \alpha_{xx} \frac{\partial T}{\partial x}, \tag{4.1}$$

$$q_x = \pi_{xx} J_x - \kappa_{xx} \frac{\partial T}{\partial x}. \tag{4.2}$$

Consider the system of Fig. 1, where element A is a p-type semiconductor with ends at temperatures $T_2 > T_1$, and connecting leads of material B going to contacts 0,3 at temperature T_0.

The charge carriers — in this case positive holes — drift down the temperature gradient to produce an excess of positive charge at the lower temperature end. If the potentials, under open circuit conditions, at the respective contacts are φ_0 and φ_3, then, for the sign convention used by Price,[23] the thermoelectric power of the couple at temperature T is defined as

$$(TEP)_{\text{couple}} \equiv \lim_{\substack{T_2 \to T \\ T_1 \to T}} \left\{ \frac{\varphi_0 - \varphi_3}{T_1 - T_2} \right\} \equiv \alpha_{AB} \equiv \alpha_B - \alpha_A. \tag{4.3}$$

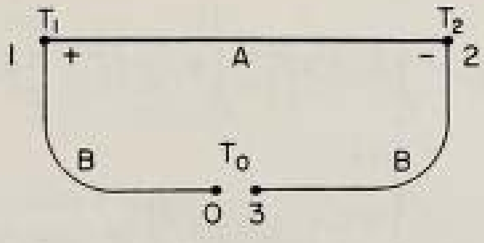

FIG. 1. Thermoelectric circuit with p-type semiconductor as element A. $T_2 = T + \Delta T/2$; $T_1 = T - \Delta T/2$.

It therefore follows that to obtain the thermoelectric power of semiconductor A at temperature T, one must determine

$$\lim_{\Delta T \to 0} \frac{\varphi_0 - \varphi_3}{\Delta T} \equiv \alpha_{BA} \equiv \alpha_A - \alpha_B \tag{4.4}$$

where element B is chosen so that α_B is known precisely or else is negligibly small.

We shall now show that (4.4) does yield the α_{xx} in Eq. (4.1). By summing potential drops in each section of the circuit of Fig. 1, one obtains

$$\varphi_0 - \varphi_3 = \tag{4.5}$$
$$= (\varphi_0 - \varphi_1)_B + (\varphi_B - \varphi_A)_{T_1} + (\varphi_1 - \varphi_2)_A + (\varphi_A - \varphi_B)_{T_2} + (\varphi_2 - \varphi_3)_B.$$

At each junction, with no electric current, the electrochemical potential Λ $(\equiv \zeta - e\varphi)$ must be continuous. Thus, with the additional use of Eq. (1.3), one obtains

$$\varphi_0 - \varphi_3 = \int\limits_{(1)}^{(0)} \left[- E^* + \frac{1}{e}\frac{\partial \zeta}{\partial x} \right]_B dx + \frac{1}{e}(\zeta_B - \zeta_A)_{T_1} + \tag{4.6}$$
$$+ \int\limits_{(2)}^{(1)} \left[- E^* + \frac{1}{e}\frac{\partial \zeta}{\partial x} \right]_A dx + \frac{1}{e}(\zeta_A - \zeta_B)_{T_2} + \int\limits_{(3)}^{(2)} \left[- E^* + \frac{1}{e}\frac{\partial \zeta}{\partial x} \right]_B dx,$$

$$\varphi_0 - \varphi_3 = \frac{1}{e}(\zeta_{B,0} - \zeta_{B,3}) + \int\limits_{(0)}^{(1)} E_B^* \, dx + \int\limits_{(1)}^{(2)} E_A^* \, dx + \int\limits_{(2)}^{(3)} E_B^* dx. \tag{4.7}$$

Since points 0 and 3 are at the same temperature, the first term vanishes, and use of (4.1) gives

$$\varphi_0 - \varphi_3 = \int\limits_{T_0}^{T_1} \alpha_{xx,B} \, dT + \int\limits_{T_1}^{T_2} \alpha_{xx,A} \, dT + \int\limits_{T_2}^{T_0} \alpha_{xx,B} \, dT \tag{4.8}$$

$$= \int\limits_{T - \Delta T/2}^{T + \Delta T/2} (\alpha_{xx,A} - \alpha_{xx,B}) \, dT. \tag{4.9}$$

Thus

$$\lim_{\Delta T \to 0} \frac{\varphi_0 - \varphi_3}{\Delta T} = \alpha_{xx,A} - \alpha_{xx,B} \qquad \text{Q.E.D.} \tag{4.10}$$

In order to examine the Peltier effect, terminals 0,3 are joined by a resistance. For this power generation arrangement, heat is absorbed at T_2 and discharged at T_1. If the coordinate system is chosen so that $(\partial T/\partial x)_A > 0$, then $J_{x,A} < 0$ and $q_{x,A}$ at each junction is negative. Also, according to our sign convention, given in Section 2, it follows that

$\pi_A - \pi_B > 0$ at either junction. Thus, since π_B of the contacting metal is small, it follows that $\pi_A > 0$. This is in agreement with the sign of $\pi_{xx,A}$ as given by (4.2).

If external power is applied to terminals 0,3 so that $J_{x,A} > 0$, then heat is absorbed at T_1 and discharged at T_2, such process being commonly referred to as Peltier cooling. Inasmuch as both $q_{x,A}$ and $J_{x,A}$ have changed signs, $\pi_{xx,A}$ is still positive, as it should be, since it is representative of material A.

b. Isotropic Case, Transverse Magnetic Field, Absence of Thermal Gradients

The transport equations (3.5) assume the form

$$\begin{aligned} J_x &= \sigma_{xx}(\mathbf{H})E_x^* + \sigma_{xy}(\mathbf{H})E_y^* \\ & \qquad\qquad\qquad\qquad \mathbf{H} = (0, 0, H_z). \qquad (4.11) \\ J_y &= -\sigma_{xy}(\mathbf{H})E_x^* + \sigma_{xx}(\mathbf{H})E_y^* \end{aligned}$$

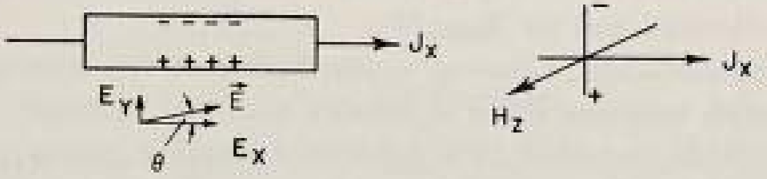

FIG. 2. Hall field due to action of magnetic field on positive charge carriers (holes). (Letters in the figures with overhead arrows are equivalent to boldface letters in the text.)

Galvanomagnetic measurements are commonly carried out under the conditions that J_y be zero. It is customary to define a Hall angle, θ, by the relation

$$\tan\theta \equiv E_y^*/E_x^* = E_y/E_x. \qquad (4.12)$$

The equality of $\mathbf{E}$ and $\mathbf{E}^*$ follows from the fact that the specimen is isothermal and homogeneous.

From (4.11) it is seen that

$$\tan\theta = \sigma_{xy}(\mathbf{H})/\sigma_{xx}(\mathbf{H}), \qquad J_y = 0. \qquad (4.13)$$

The Hall angle is simply illustrated by the free-particle example shown in Fig. 2.

Consider the motion of positive charges having velocities in the positive direction due to the impressed electric field E_x. In the absence of any transverse electric field, the Lorentz force due to the positive magnetic field vector H_z (out of the page, in Fig. 2) would produce a deflection in the negative y-direction. Therefore, to satisfy the boundary condition that J_y vanish, a transverse electric field must exist. This *Hall field* assumes such a value that the force on a charge carrier having a certain "average" velocity is zero in the crossed electric and magnetic fields. If all the charge carriers had the same velocity, then the Hall field force would exactly cancel the force due to the magnetic field on each moving charge. Hence no charge carriers would be deflected, and there would be no magnetoresistance. This is approximately the situation in an isotropic degenerate metal.[31] Here only those electrons of energy near ζ, where ζ is the Fermi energy, contribute to the conduction. In this case, as is shown in Section 14, the transverse magnetoresistance coefficient is proportional to $(d\tau/d\varepsilon)^2_{\varepsilon=\zeta}$, where τ is the relaxation time and ε the energy of the charge carriers. Thus the one-band model yields a very low magnetoresistance for isotropic metals. This point has been discussed by Coldwell-Horsfall and ter Haar.[31a]

In semiconductors, however, a distribution of velocities exists. The carriers with velocities larger or smaller than the "average" referred to above may be regarded pictorially as traversing longer paths, thus increasing the resistance of the conductor. It also follows that any mechanism which tends to short out the Hall field — for example, shorting contacts, inhomogeneities in the material, etc. — will increase the magnetoresistance. More will be said of this point later.

Continuing a step further with our pictorial representation, the equality of transverse electric and magnetic forces on the charge carriers for the case of constant velocity, yields the result that

$$E_y = v_x H_z/c. \tag{4.14}$$

Gaussian units are implied, with c the velocity of light.

Since, under the same restrictions,

$$J_x = nqv_x = \sigma_{xx} E_x \tag{4.15}$$

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[31] A. Sommerfeld and N. H. Frank, *Revs. Modern Phys.* **3**, 1 (1931).

[31a] R. Coldwell-Horsfall and D. ter Haar, *Phil. Mag.* **46**, 1149 (1955). There is a misprint in Eq. (2); the factor $\frac{1}{2}$ should be a 2.

where q is the charge on the carrier and n the concentration, it follows that

$$E_y/E_x = \sigma_{xx} H/nqc, \qquad H \equiv H_z, \tag{4.16}$$

$$\sigma_{xy} = (\sigma_{xx}{}^2/nqc)H. \tag{4.17}$$

Also, since the Hall constant R is defined by the relation

$$E_y = RJ_x H \tag{4.18}$$

it is apparent that

$$R = 1/nqc. \tag{4.19}$$

Although the above relations were established here using an extremely restricted model, they can be shown rather simply to hold for the more general case of isotropic scattering of the charge carriers characterized by a relaxation time independent of the velocity of the carriers.[32, 33]

It is apparent from Fig. 2, as well as from Eq. (4.17), for a magnetic field vector in the positive z-direction in isotropic media that[34]

$$\begin{aligned} \sigma_{xy}(\mathbf{H}) &> 0 \qquad \text{for holes,} \\ \sigma_{xy}(\mathbf{H}) &< 0 \qquad \text{for electrons.} \end{aligned} \tag{4.20}$$

It is also clear that $\sigma_{xx}(\mathbf{H}) > 0$ for either type of conduction.

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[32] William Shockley, "Electrons and Holes in Semiconductors," p. 208. Van Nostrand, New York, 1950.

[33] Harvey Brooks, Advances in Electronics and Electron Phys. 7, 128 (1955).

[34] The relations may not necessarily hold if anisotropy is present in the energy surfaces (surfaces of constant energy in momentum space). Re-entrant energy surfaces will reverse the signs in (4.20). See p. 340 of reference 32.

III.
Expressions for Charge-Carrier Transport Coefficients

5. GENERAL THEORY — THE BOLTZMANN EQUATION

The classical approach, as set forth by Lorentz and Sommerfeld, involves the determination of the distribution of the charge carriers among the different states in the presence of applied electric, magnetic, and thermal fields. If the states are expressed by the wave vectors $\mathbf{k}$, then the number of charge carriers in volume element d^3r ($\equiv dx\,dy\,dz$) characterized by their wave vectors $\mathbf{k}$ in the element d^3k ($\equiv dk_x\,dk_y\,dk_z$) is given by

$$\frac{1}{4\pi^3}\,f(\mathbf{k},\mathbf{r})\,d^3k\,d^3r \tag{5.0}$$

where $f(\mathbf{k},\mathbf{r})$ is the distribution function for the charge carriers — that is, the probability that a carrier exists in the state designated by wave vector $\mathbf{k}$ and at position $\mathbf{r}$. The above expression leads to a density of states term, given by

$$n(\varepsilon)\Delta\varepsilon = \frac{2}{(2\pi)^3}\int d^3k. \tag{5.0a}$$

The term on the left is the number of energy levels per unit volume of configuration space lying in the range ε, $\varepsilon + \Delta\varepsilon$. The integral is taken over the volume in k-space lying between the surfaces of constant energy ε and $\varepsilon + \Delta\varepsilon$. The factor of 2 is included because of spin, and the $(2\pi)^{-3}$ enters as a result of the use of the wave vector $\mathbf{k}$, to designate the quantum mechanical states.[34a,b]

[34a] See pp. 15 and 43 of Wilson's book.[9]

[34b] See pp. 55–57 *in* N. F. Mott and H. Jones, "The Theory of the Properties of Metals and Alloys," Oxford Univ. Press, London and New York, 1936.

In the classical limit (free electrons represented by plane waves) the momentum is related to the wave vector by the expression $\mathbf{p} = \hbar\mathbf{k}$, and it is seen that Eq. (5.0a) is equivalent to saying that two quantum states (including each direction of electron spin) exist in each volume h^3 in phase space,[34c] i.e.,

$$n(\varepsilon)\Delta\varepsilon = \frac{2}{h^3}\int d^3p.$$

Once $f(\mathbf{k}, \mathbf{r})$ is known, the electrical and thermal currents due to the charge carriers are given by

$$\mathbf{J} = \left(-\frac{e}{4\pi^3}\right)\int \mathbf{v} f\, d^3k \tag{5.1}$$

$e > 0$ for electrons

$$\mathbf{q}_{el} = \left(\frac{1}{4\pi^3}\right)\int \mathbf{v}(\varepsilon - \zeta) f\, d^3k \tag{5.2}$$

where $\mathbf{v}$ is the velocity of the charge carrier, and $\varepsilon - \zeta$ is its energy above the Fermi energy.

In the absence of fields, the function $f(\mathbf{k}, \mathbf{r})$ becomes the Fermi-Dirac distribution $f_0(\varepsilon)$, namely,

$$f_0(\varepsilon) = \frac{1}{e^{(\varepsilon - \zeta)/kT} + 1}. \tag{5.3}$$

When fields exist, the usual procedure is to determine f by obtaining a solution of the Boltzmann equation. An alternative procedure, which has been employed in special cases, is a pictorial kinetic method based on calculations of the average drift velocity in the direction of the applied electric field. Certain difficulties which can arise in such an approach are discussed by Dingle.[34d] To overcome these, he introduces a *transport distribution function* defined, for a drift velocity in the z-direction, by

$$f_t = -v_z\, \partial f_0/\partial v_z. \tag{5.3a}$$

[34c] See, for example, p. 335 of the article by A. Sommerfeld and H. Bethe in "Handbuch der Physik" (H. Geiger and Karl Scheel, eds.), Vol. 24, Part 2, p. 333. Springer, Berlin, 1933.

[34d] R. B. Dingle, *Physica* 22, 671 (1956).

The use of this transport distribution function in obtaining results valid for semiconductors ($\zeta/kT \ll 0$, where ζ is positive when measured upward from the bottom of the conduction band), from relationships derived for metals ($\zeta/kT \gg 1$)[34e] is illustrated.

The Boltzmann equation referred to in the preceding paragraph is an expression of the fact that at steady state df/dt must vanish. The total rate of change of f is expressed as the sum of a term due to the applied fields and one due to collisions of the charge carriers with other particles, imperfections, or lattice vibrations:

$$(\partial f/\partial t)_{\text{fields}} + (\partial f/\partial t)_{\text{coll}} = 0. \tag{5.4}$$

The first factor is commonly expressed in terms of applied fields and the gradients of f as indicated below:

$$-\frac{e}{\hbar}\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{H}\right) \cdot \text{grad}_{\mathbf{k}} f + \mathbf{v} \cdot \text{grad}_{\mathbf{r}} f = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}, \tag{5.5}$$

$e > 0$ for electrons.

The second term in the above equation is concerned with spatial variation in f, and in homogeneous materials it enters because of temperature gradients.

In the case of perfectly free electrons, the representation of $(\partial f/\partial t)_{\text{fields}}$ in terms of the electric and magnetic field vectors is precisely given by the first term of (5.5). The more realistic case of electrons moving in a periodic potential was studied by Jones and Zener.[35, 36] More recently, Adams and Argyres[37] have also considered certain aspects of the problem. These investigations have established for the periodic lattice, that starting with (5.4), the expression (5.5) is valid to first order in **E** and for values of **H** and times sufficiently short, so that $eHt/m^* c \ll 1$. The quantity m^*, which is the "effective mass" of the electron in the periodic potential of the solid, is discussed in more detail in Section 6*a*. Another consideration

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[34e] The symbol "k" in the terms kT here, as well as in (5.3) is of course the Boltzmann constant, not the magnitude of the wave vector.

[35] H. Jones and C. Zener, *Proc. Roy. Soc.* **A144**, 101 (1934).

[36] Reference 9, p. 45 ff.

[37] E. N. Adams and P. N. Argyres, *Phys. Rev.* **102**, 605 (1956).

which also requires that the accelerations be not too large is that interband mixing of states does not occur.[37]

The condition obtained from Jones and Zener's derivation, namely, that $eHt/m^* c \ll 1$, appears to be more restrictive than is necessary. In fact, as will be seen later in discussing measurements of the Hall effect as a function of magnetic field in germanium and in indium antimonide, conductivity coefficients based on the Boltzmann equation seem to be reliable for values of $eHt/m^* c$ of 10^2 or 10^3, where t here is interpreted as the relaxation time. The principal restriction in the case of these high-field measurements is that the energy separation between quantized magnetic levels be significantly less than the Fermi energy, or than kT in the case of nondegenerate carriers, that is,

$\hbar\omega \ll \zeta$ (degenerate charge carriers), or (5.6)

$\hbar\omega \ll kT$ (classical statistics).

In the above, k is the Boltzmann constant, T the absolute temperature, and ω ($\equiv eH/m^* c$) is the cycltron frequency. If condition (5.6) does not hold, then orbit quantization of the charge carriers becomes important. These strong-field effects are discussed briefly in Section 28. For further comments on the Jones-Zener conditions, the reader is referred to a discussion by Chambers.[38] Several articles by Kohn deal with the theory of Bloch electrons in a magnetic field.[38a] Pertinent comments also appear in the treatise by Ziman.[38b] See also Section 28*a*, namely footnote 498d.

The Boltzmann transport equation is, of course, a classical equation. Recently, Van Hove,[39] Kohn and Luttinger,[40] and others have presented quantum theories of transport. Special treatments, also taking into account quantum effects, have been developed by a number of investigators for the strong-magnetic-field limit. These cases will be discussed in more detail later (Section 28).

Kohn and Luttinger[40] pointed out a number of weaknesses in a conductivity theory based on relationships such as Eq. (5.4). These include

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[38] R. G. Chambers, *Proc. Roy. Soc.* **A238**, 344 (1957).

[38a] W. Kohn, *Proc. Phys. Soc. (London)* **72**, 1147 (1958); *Phys. Rev.* **115**, 1460 (1959).

[38b] J. M. Ziman, "Electrons and Phonons." Oxford Univ. (Clarendon) Press, London and New York, 1960. In particular pp. 96 and 512.

[39] Leon Van Hove, *Physica* **21**, 517 (1955).

[40] W. Kohn and J. M. Luttinger, *Phys. Rev.* **108**, 590 (1957).

the random-phase assumption discussed by Van Hove[39] in his investigations of the approach to equilibrium of a quantum many-body system,[40a, b] and the fact that in the usual form of (5.4) the collision interaction is treated by the lowest order of perturbation theory.

Also, in the usual Boltzmann equation, the rate of change of the distribution function f at a given time depends only on the value of f at that time. In the more general case, the rate of change of f depends on all previous values up to the time in question. A number of papers dealing with the Boltzmann equation and with the transport of energy and matter in nonequilibrium systems were presented at the Conference on Transport Processes in Statistical Mechanics in 1956 in Brussels.[40c]

In Kohn and Luttinger's quantum mechanical development,[40] a density matrix[40d] formulation of the problem is used. Expansions are obtained in ascending powers of the strength of the scattering potential of "random" rigid impurity centers. In the limiting case of very weak interactions causing collisions, results reduce to the standard Boltzmann equation. In a subsequent treatment, the same investigators[41] develop the transport equation in powers of the density of scattering centers, without restriction to weak scattering potentials. Again, the first-order terms yield the Boltzmann equation. The density matrix approach has also been used by Lifshitz[42] in his development of a quantum theory of the electrical conductivity in metals in magnetic fields. Expressions are given for the asymptotic behavior of the conductivity tensor in strong magnetic fields — i.e., $\omega\tau \gg 1$ — a region where results could be deduced without special assumptions about the collision integral. These results are given as series in the variable $1/\omega t_0$, where t_0 is a relaxation time. Comparisons are made between the quantum and the classical expressions. Although the treatment by Lifshitz includes oscillatory effects, the region of the "quantum limit" effects in semiconductors (see Section 28) is excluded in that the theory involves the assumption $\hbar\omega \ll \zeta$.

[40a] Leon Van Hove, *Physica* **25**, 268 (1959).

[40b] L. Van Hove and E. Verboven, *Physica* **27**, 418 (1961).

[40c] Proceedings of the International Symposium on "Transport Processes in Statistical Mechanics" (I. Prigogine, ed.). Interscience, New York, 1958.

[40d] See, for example, D. ter Haar, "Elements of Statistical Mechanics," p. 147. Rinehart, New York, 1954.

[41] J. M. Luttinger and W. Kohn, *Phys. Rev.* **109**, 1892 (1958).

[42] I. M. Lifshitz, *Phys. and Chem. Solids* **4**, 11 (1958).

Since the solution of the Boltzmann equation is discussed rather extensively in Wilson[9] and also by Fan[43] and Blatt,[44] only a short discussion is given here. The general treatment involves putting

$$f = f_0 - \phi(\mathbf{k})\,(\partial f_0/\partial \varepsilon). \tag{5.7}$$

To first order in ϕ, the collision term is given by

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = -\frac{1}{kT}\int V(\mathbf{k},\mathbf{k}')\,\{\phi(\mathbf{k}) - \phi(\mathbf{k}')\}\,d^3k' \tag{5.8}$$

where

$$V(\mathbf{k}, \mathbf{k}') \equiv \mathscr{W}(\mathbf{k}, \mathbf{k}')f_0(\mathbf{k})\{1 - f_0(\mathbf{k}')\} = V(\mathbf{k}', \mathbf{k}) \tag{5.9}$$

and $\mathscr{W}(\mathbf{k}, \mathbf{k}')$ is the probability per unit time that an electron makes a transition from state $\mathbf{k}$ to state $\mathbf{k}'$.

The solution of (5.5) with (5.8) is somewhat involved and is usually accomplished by variational methods (see Section 25). The problem is greatly simplified when a relaxation time can be introduced such that

$$(\partial f/\partial t)_{\text{coll}} = -(f - f_0)/\tau. \tag{5.10}$$

We shall discuss this case in the next section, returning later (Section 25) to the more general approach.

6. USE OF A RELAXATION TIME IN THE BOLTZMANN EQUATION

When the approach of the distribution function f to equilibrium, after the external fields are removed, can be expressed by

$$(f - f_0)_t = (f - f_0)_{t=0}\,e^{-t/\tau}, \tag{6.1}$$

then the integral equation (5.8) reduces to the simple relationship (5.10). With this simplification, the Boltzmann equation can quite readily be solved for a number of conditions of interest. In most cases, the relaxation time τ is considered as a function of the energy only of the charge carriers,

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[43] H. Y. Fan, *Solid State Phys.* 1, 283 (1955).

[44] F. J. Blatt, *Solid State Phys.* 4, 199 (1957).

that is, it is isotropic on a surface of constant energy. If the energy surfaces have spherical symmetry, then τ is a function only of $|\mathbf{k}|$. In certain anisotropic solids, directional effects may be so important that it is desirable to take into account $\tau(\mathbf{k})$.

The use of a relaxation time appears quite justified for elemental semiconductors such as germanium and silicon, except perhaps at very low temperatures. An important condition, pointed out by Howarth and Sondheimer,[45] is that the energy emitted or absorbed by a charge carrier at collision must be small compared with its initial energy (see also Section 25*b*).

With a relaxation time, the expressions for f, to a first order, are relatively simple. It is convenient to distinguish two cases.

a. Quadratic Energy Surfaces

When the energy of the charge carriers is a quadratic function of their wave number it is possible to obtain an exact solution of the Boltzmann equation and to determine the distribution function f to first order in ϕ [see Eq. (5.7)] for small electric and thermal gradients.

A general quadratic expression, representing an expansion of the energy to second degree terms in $\mathbf{k}$ about an energy minimum at $\mathbf{k}_0$ may be written

$$\mathscr{E}(\mathbf{k}) = \mathscr{E}(\mathbf{k}_0) + (\hbar^2/2)(\mathsf{M}^{-1})_{ij}(k_i - k_{0i})(k_j - k_{0j}). \qquad (6.1\text{a})$$

Such an expansion is possible if the energy bands are not degenerate at the minima.[45a] Also there must be no combination of spin-orbit splitting and lack of inversion symmetry in the solid so that energy terms linear in k can occur.[45b,c] In cases involving such energy surfaces, the concept of effective mass does not exist in the usual sense, as is considered below.

The tensor M^{-1} in (6.1a) is commonly termed the inverse mass tensor or reciprocal mass tensor, and its components are

$$(\mathsf{M}^{-1})_{ij} = (1/\hbar^2)\, \partial^2\mathscr{E}(k)/\partial k_i\, \partial k_j. \qquad (6.1\text{b})$$

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[45] D. J. Howarth and E. H. Sondheimer, *Proc. Roy. Soc.* **A219**, 53 (1953).

[45a] See, for example, the discussion on pp. 148 and 177 in the treatise, G. H. Wannier, "Elements of Solid State Theory," Cambridge Univ. Press, London and New York, 1959.

[45b] R. H. Parmenter, *Phys. Rev.* **100**, 573 (1955).

[45c] G. Dresselhaus, *Phys. Rev.* **100**, 580 (1955).

For spherical energy surfaces with the energy minimum at $\mathbf{k} = 0$, the reciprocal mass tensor reduces to a scalar, commonly known as the "effective mass," m*:

$$(\mathsf{M}^{-1})_{ij} = (1/m^*)\,\delta_{ij}. \tag{6.1c}$$

The effective mass differs from the free mass of the charge carrier because of the potential energy in the crystal.[45d] If this were constant, then m^* would be identical with the free electron mass m_0. When one considers the relation between acceleration and force on an electron in a periodic potential, it turns out that an electron in an energy band behaves in first approximation as though it had an effective mass $\mathbf{m}^*$ represented by the tensor

$$(\mathbf{m}^*)^{-1} = (1/\hbar^2)\,\mathrm{grad}_{\mathbf{k}}\,\mathrm{grad}_{\mathbf{k}}\,\mathscr{E}(\mathbf{k}), \tag{6.1d}$$

an expression which is analogous to (6.1b). An equivalent way of expressing the difference between m_0 and m^*, as pointed out by Kittel[45e], is to note that the latter takes into account the interaction of the electron and the crystal lattice. For example, in the case of negative m^*, the momentum transfer to the lattice is opposite to and larger than that transferred to the electron. That is, the Bragg reflections result in an actual decrease in the momentum of the electron.

Where possible, one usually chooses a coordinate system such that the inverse mass tensor is reduced to principal axes, thus[45f]

$$(\mathsf{M}^{-1})_{ij} = (1/m_i^*)\,\delta_{ij}, \tag{6.1e}$$

where we have followed the customary policy of abbreviating m_{ii}^* by m_i^*, it being implied that these are the components of the diagonalized mass tensor. For this specialized coordinate system, the quadratic relationship between energy and wave number becomes

$$\mathscr{E}(\mathbf{k}) = \mathscr{E}(\mathbf{k}_0) + (\hbar^2/2)(1/m_i^*)(k_i - k_{0i})^2. \tag{6.1f}$$

[45d] See pp. 141 and 317 in the treatise by F. Seitz.[209]

[45e] C. Kittel, *Am. J. Phys.* 22, 250 (1954).

[45f] It is obvious that in an equation where indices are repeated on both sides, the convention of summation over such indices does not apply.

Finally, if the energy zero is at the minimum and if this point is taken as the origin of the coordinate system, i.e., $k_0 = 0$, then (6.1f) assumes a form which is frequently found in the literature, namely,

$$\varepsilon(k) = \tfrac{1}{2}\hbar^2[k_1{}^2/m_1{}^* + k_2{}^2/m_2{}^* + k_3{}^2/m_3{}^*]. \tag{6.2}$$

The expression which is commonly quoted for the distribution function in the general case of quadratic energy surfaces is due to Bronstein[46] or to Blochinzev and Nordheim[47]. It was also developed by Jones[48] for application to bismuth. The expression for $\phi(\mathbf{k})$ when τ is a function of energy only is given in a number of review articles such as those by Fan[43] and Blatt[44], and may be written in the form

$$\phi(k) = -\frac{e\tau}{\hbar}\{\text{grad}_{\mathbf{k}}\,\varepsilon\}\cdot\left\{\frac{\mathbf{F} - (e\tau/c)\mathsf{M}^{-1}\mathbf{F}\times\mathbf{H} + (e\tau/c)^2(\mathbf{F}\cdot\mathbf{H}\mathsf{M}\mathbf{H}/||\mathsf{M}||}{1 + (e\tau/c)^2(\mathsf{M}\mathbf{H}\cdot\mathbf{H}/||\mathsf{M}||)}\right\}. \tag{6.2a}$$

The quantity M is the effective mass tensor and $||\mathsf{M}||$ is its determinant. For the case of spherical energy surfaces, the mass tensor reduces to a scalar, m^*, and (6.2a) becomes

$$\phi(\mathbf{k}) = -\frac{e\tau}{\hbar}\{\text{grad}_{\mathbf{k}}\,\varepsilon\}\cdot\left\{\frac{\mathbf{F} - (e\tau/m^*\,c)\mathbf{F}\times\mathbf{H} + (e\tau/m^*\,c)^2\,\mathbf{F}\cdot\mathbf{H}\mathbf{H}}{1 + (e\tau/m^*\,c)^2\,H^2}\right\} \tag{6.3}$$

where our sign convention is such that $e > 0$ for electrons, and where $\mathbf{F}$, the applied electrothermal field, is related to the $\mathbf{P}$ of Wilson[49] as follows:

$$\mathbf{P} \equiv -e\mathbf{F} \equiv -e\mathbf{E} + T\,\text{grad}_{\mathbf{r}}\,[(\varepsilon - \zeta)/T]. \tag{6.4}$$

In terms of the gradients used in Chapter II, the expression becomes

$$\mathbf{F} \equiv \mathbf{E}^* + \frac{1}{eT}(\varepsilon - \zeta)\,\text{grad}_{\mathbf{r}}\,T. \tag{6.5}$$

The electrical and thermal current densities due to the charge carriers can be written in terms of ϕ as follows:

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[46] M. Bronstein, *Physik. Z. Sowjetunion* **2**, 28 (1932).

[47] D. Blochinzev and L. Nordheim, *Z. Physik* **84**, 168 (1933).

[48] H. Jones, *Proc. Roy. Soc.* **A155**, 653 (1936).

[49] Reference 9, p. 224.

$$\mathbf{J} = \frac{e}{4\pi^3}\int \mathbf{v}\phi\,\frac{\partial f_0}{\partial\varepsilon}\,d^3k = \frac{e}{4\pi^3\hbar}\int \mathrm{grad}_{\mathbf{k}}\,\varepsilon\,\phi\,\frac{\partial f_0}{\partial\varepsilon}\,d^3k \qquad e>0 \text{ for electrons,} \tag{6.6}$$

$$\mathbf{q}^{el} = \frac{-1}{4\pi^3}\int \mathbf{v}(\varepsilon-\zeta)\phi\,\frac{\partial f_0}{\partial\varepsilon}\,d^3k = \frac{-1}{4\pi^3\hbar}\int \mathrm{grad}_{\mathbf{k}}\,\varepsilon\,(\varepsilon-\zeta)\,\phi\,\frac{\partial f_0}{\partial\varepsilon}\,d^3k. \tag{6.7}$$

These results follow from (5.1) and (5.2) with the use of (5.7). Use is also made of the relation

$$\hbar\mathbf{v} = \mathrm{grad}_{\mathbf{k}}\,\varepsilon \tag{6.8}$$

The case of ellipsoidal energy surfaces can also be treated from transport coefficients based on (6.3), since one can make a linear transformation in **k**-space which transforms the ellipsoids to spheres.[50,51] This approach may at times be preferable to the direct use of the expression for f for quadratic energy surfaces.

b. Nonquadratic Energy Surfaces

In the preceding case, an exact solution of the Boltzmann equation was possible, so that results were valid for arbitrary values of H — subject, of course, to the restrictions on (5.5), which were discussed. When, however, the energy of the charge carriers is not a quadratic function of their wave numbers, then it is necessary to resort to series solutions of the Boltzmann equation. Such a solution was obtained for the isothermal case by Jones and Zener[52] in ascending powers of H.[49] The results can also be expressed in terms of the electrothermal field vector, **F**, Eq. (6.5). The procedure is to write the Boltzmann equation in terms of an operator **Ω**, to be discussed later, as follows:

$$\frac{1}{\tau}\phi + \frac{e}{\hbar}\mathbf{F}\cdot\mathrm{grad}_{\mathbf{k}}\,\varepsilon + \frac{e}{\hbar^2 c}\mathbf{H}\cdot\mathbf{\Omega}\phi = 0. \tag{6.9}$$

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[50] Reference 44, p. 285.

[51] E. H. Sondheimer, *Proc. Roy. Soc.* **A224**, 260 (1954).

[52] H. Jones and C. Zener, *Proc. Roy. Soc.* **A145**, 268 (1934).

This expression follows from (5.5) with the use of (5.7), (5.10), (6.5), and (6.8) with neglect of terms involving products of $\mathbf{F}$ and ϕ. Equation (6.9) is now solved by an iterative method to yield

$$\phi = \frac{-e\tau}{\hbar}\left\{\mathbf{F}\cdot\mathrm{grad}_{\mathbf{k}}\,\varepsilon - \left(\frac{e}{\hbar^2 c}\right)\mathbf{H}\cdot\mathbf{\Omega}(\tau\mathbf{F}\cdot\mathrm{grad}_{\mathbf{k}}\,\varepsilon) + \right.$$
$$\left. + \left(\frac{e}{\hbar^2 c}\right)^2\mathbf{H}\cdot\mathbf{\Omega}[\tau\mathbf{H}\cdot\mathbf{\Omega}(\tau\mathbf{F}\cdot\mathrm{grad}_{\mathbf{k}}\,\varepsilon)] + \ldots\right\}. \tag{6.10}$$

This expansion is valid when

$$e\tau H/m^* c \equiv \omega\tau < 1. \tag{6.10a}$$

The $\mathbf{\Omega}$ is an operator, usually written as

$$\mathbf{\Omega} \equiv \mathrm{grad}_{\mathbf{k}}\,\varepsilon \times \mathrm{grad}_{\mathbf{k}} \tag{6.11}$$

or in tensor notation

$$\Omega_i \equiv \varepsilon_{ijl}\frac{\partial\varepsilon}{\partial k_j}\frac{\partial}{\partial k_l}. \tag{6.12}$$

Summation over repeated indices is implied, and the permutation tensor ε_{ijl} — not to be confused with the energy ε — is defined in the usual manner,

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \qquad \varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1 \tag{6.13}$$

with all other components being zero.

In evaluating (6.9), it is helpful to note that $\mathbf{\Omega}$ commutes with any parameter which is a function of energy only; that is if $F(\varepsilon)$ is an arbitrary function of $\varepsilon(\mathbf{k})$, then

$$\mathbf{\Omega}F = F\mathbf{\Omega}. \tag{6.14}$$

This is easily established by operating on the function $F(\varepsilon)G(\mathbf{k})$ with Ω_i. Thus,

$$\Omega_i F(\varepsilon)G(\mathbf{k}) = \varepsilon_{ijl}\frac{\partial\varepsilon}{\partial k_j}\left[G(\mathbf{k})\,\frac{\partial F}{\partial\varepsilon}\,\frac{\partial\varepsilon}{\partial k_l} + F\,\frac{\partial}{\partial k_l}\,G(\mathbf{k})\right]$$
$$= F\varepsilon_{ijl}\frac{\partial\varepsilon}{\partial k_j}\frac{\partial}{\partial k_l}G = F\Omega_i G,$$

since it is readily apparent that

$$\varepsilon_{ijl}\frac{\partial\varepsilon}{\partial k_j}\frac{\partial\varepsilon}{\partial k_l} \equiv 0,$$

or in vector form,

$$(\mathrm{grad}_{\mathbf{k}}\,\varepsilon \times \mathrm{grad}_{\mathbf{k}}\,\varepsilon)_i \equiv 0.$$

Expressions for the conductivity coefficients in ascending powers of H are adequate for studying weak-magnetic-field phenomena but are not very useful when it is desired to investigate the magnetic field dependence of galvanomagnetic phenomena. Approximately at the point where the experimental results become interesting, the power series in H converges extremely slowly, or even begins to diverge. In such cases it may be advantageous to use a method discussed by Shockley[53] and developed in more detail by Chambers[38] and by McClure.[54]

In McClure's development, the components of the conductivity tensor are expressed as Fourier series expansions in harmonics of the frequency of the charge carrier around the hodograph in the magnetic field. This hodograph is the curve in k-space formed by the intersection of a surface of constant energy with a plane perpendicular to the magnetic field. In particular with $\mathbf{H}$ along the k_z-direction, the hodograph lies in the $k_x\,k_y$-plane, i.e., where k_z is constant. The third term of the Boltzmann equation, (6.9), expressed in terms of the velocity of the charge carrier, $\mathbf{v} = \hbar^{-1}\,\mathrm{grad}_{\mathbf{k}}\,\varepsilon$, is

$$(e/\hbar c)\mathbf{H} \times \mathbf{v} \cdot \mathrm{grad}_{\mathbf{k}}\,\phi. \tag{6.15}$$

This may be written as

$$\left(\frac{e}{\hbar c}\right) H v_{\perp} \frac{\partial\phi}{\partial k_s} \tag{6.16}$$

where $v_{\perp}$ is the magnitude of the velocity component perpendicular to $\mathbf{H}$, and $d\mathbf{k}_s$ is the component of the differential wave vector along the direction $\mathbf{v} \times \mathbf{H}$. Thus dk_s is an element of arc along the hodograph. Let us

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[53] W. Shockley, *Phys. Rev.* **79**, 191 (1950).

[54] J. W. McClure, *Phys. Rev.* **101**, 1642 (1956).

denote by $t_s(\mathbf{k})$ the time at which an electron, precessing around the hodograph, in the absence of the electric field, is at point $\mathbf{k}$. Then the classical equation of motion on the hodograph yields

$$\frac{\partial(\hbar k_s)}{\partial t_s} = \left(\frac{e}{c}\right) H v_\perp \tag{6.17}$$

where magnitude of the momentum on the hodograph has been expressed as $\hbar k_s$. Thus the term (6.15) is simply $\partial\phi/\partial t_s$, and the Boltzmann equation takes the form

$$\partial\phi/\partial t_s + \phi/\tau + e\mathbf{E}\cdot\mathbf{v} = 0. \tag{6.18}$$

Although here we have included only the electric field, results are readily carried through for the electrothermal field vector $\mathbf{F}$. Solution of the above first-order differential equation is straightforward. The constant of integration is determined by the condition that ϕ be a single-valued function of $\mathbf{k}$, i.e., it must be periodic in t_s with a period equal to T, the time for the charge carrier to go around the hodograph. Then, as can be seen from (6.18), it follows that $\mathbf{v}$ and $\boldsymbol{\tau}$ are also periodic functions of t_s with the same period. The expression for T, and the cyclotron frequency ω, is

$$T \equiv \frac{2\pi}{\omega} = \frac{\hbar c}{eH} \oint dk_s/v_\perp. \tag{6.19}$$

The cyclotron frequency ω is related to a mass parameter m_H, namely,

$$\omega = eH/m_H c. \tag{6.19a}$$

In the case of spherical energy surfaces, m_H is identical with the effective mass m^*. In the general case, however, it will involve the components of the effective mass tensor and will depend on the orientation of the magnetic field relative to the energy surfaces[54a, b]

Because of the periodicity of $\mathbf{v}$, it can be written as a Fourier series,

$$\mathbf{v} = \sum_{m=-\infty}^{\infty} \mathbf{v}(m) e^{im\omega t_s}. \tag{6.20}$$

[54a] See, for example, p. 199 of the text by Wannier.[45a]

[54b] See, p. 514 of the text by Ziman.[38b]

Using (6.20), and the assumption that τ is constant on the hodograph, McClure is able to write the solution of (6.18) as follows:

$$\phi = -e\tau\mathbf{E}\cdot \sum_{m=-\infty}^{\infty} \frac{\mathbf{v}(m)e^{im\omega t_s}}{1+im\omega\tau}. \tag{6.21}$$

In determining the current density, the quantity $\phi\mathbf{v}(\mathbf{k})$ is first averaged over the hodograph which passes through the point $\mathbf{k}$,

$$\begin{aligned} \langle\phi\mathbf{v}\rangle_s &= \frac{1}{T}\oint_0^T \phi\mathbf{v}\,dt_s \qquad (6.22) \\ &= -\frac{e\tau}{T}\mathbf{E}\cdot\oint_0^T \sum_{m=-\infty}^{\infty} \frac{\mathbf{v}(m)e^{im\omega t_s}}{1+im\omega\tau} \sum_{n=-\infty}^{\infty} \mathbf{v}(n)e^{in\omega t_s}\,dt_s. \end{aligned}$$

Because of the orthogonality of the exponentials, the above reduces to

$$\langle\phi\mathbf{v}\rangle_s = -e\tau\mathbf{E}\cdot \sum_{m=-\infty}^{\infty} \frac{\mathbf{v}(m)\mathbf{v}(-m)}{1+im\omega\tau}. \tag{6.23}$$

Here the magnetic field enters through the frequency ω, which follows from (6.19) once the shape of the energy surfaces is known. The Fourier components are determined from the appropriate gradients of the energy. Hence in principle, if $\varepsilon(\mathbf{k})$ is known, the problem is solved. There is, of course, the requirement that the relaxation time be a function of energy only, a restriction which was assumed in the derivation of (6.23).

Since for complex energy surfaces, analytical determinations of the Fourier components of the velocity can be very tedious, most evaluations have involved specialized cases. Of course, where the surfaces are spheres, the sequence of components terminates with $m = \pm 1$. Also, when the surfaces can be approximated by a cube, exact calculations are readily possible.[55] Where the band is a warped sphere, in particular the heavy mass valence band in germanium, Beer and Willardson[56] have obtained

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[55] C. Goldberg, E. Adams, and R. Davis, *Phys. Rev.* **105**, 865 (1957).

[56] A. C. Beer and R. K. Willardson, *Phys. Rev.* **110**, 1286 (1958).

good agreement with experiment by considering only two harmonics in addition to the $m = \pm 1$ terms. Because of the cubic symmetry, only terms with m odd occur. These two special cases are discussed in detail later (Section 21).

The question of closed and open orbits in k-space formed by the intersection of a surface of constant energy, namely, the Fermi surface, with a plane normal to the direction of the magnetic field has been discussed by a number of authors[38,54b,57] in investigations of the galvanomagnetic effects in metals (see also Section 26). Ziman[58] has considered cylindrical Fermi surfaces — another example which can be treated exactly — with axes in certain principal directions. In order to compare results with some existing experimental data taken on polycrystalline specimens, the conductivity tensor is averaged over all orientations of the crystal axes relative to the magnetic field.

7. Expressions for Transport Coefficients in Isotropic Solids

a. Conductivity Coefficients

As a result of the isotropy, the transport tensors simplify greatly as was seen in Section 3. Concise expressions can be obtained for the diagonal elements, which are even functions of H. The same is true for the off-diagonal elements, which are odd in H, and therefore vanish when the magnetic field is zero.

In order to develop the explicit expressions we shall rewrite (6.6) and (6.7) using the tensor notation implying summations over repeated indices through the three coordinate directions:

$$J_i = \frac{e}{4\pi^3 \hbar} \int \frac{\partial \varepsilon}{\partial k_i} \phi \frac{\partial f_0}{\partial \varepsilon} d^3k \qquad e > 0 \text{ for electrons} \tag{7.1}$$

$$q_i^{(el)} = -\frac{1}{4\pi^3 \hbar} \int \phi(\varepsilon - \zeta) \frac{\partial \varepsilon}{\partial k_i} \frac{\partial f_0}{\partial \varepsilon} d^3k \tag{7.2}$$

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[57] I. M. Lifshitz, M. Ia. Azbel', and M. I. Kaganov, *J. Exptl. Theoret. Phys. (USSR)* **31**, 63 (1956) [translation: *Soviet Phys.–JETP* **4**, 41 (1957)].

[58] J. M. Ziman, *Phil. Mag.* [8] **3**, 1117 (1958).

where the superscript on the heat current denotes that contribution due to the mobile charge carriers. This distinction is made since it is necessary to consider the heat transfer by other processes, for example by the lattice phonons, to obtain the total thermal current.

Making use of the permutation tensor ε_{ijl}, one can write Eq. (6.3) as follows:

$$\phi(\mathbf{k}) = -\frac{e\tau}{\hbar}\frac{1}{1+(e\tau/m^* c)^2 H^2}\cdot \left\{\left[\frac{\partial\varepsilon}{\partial k_j} - \frac{e\tau}{m^* c}\varepsilon_{ljm}\frac{\partial\varepsilon}{\partial k_l}H_m + \left(\frac{e\tau}{m^* c}\right)^2\frac{\partial\varepsilon}{\partial k_l}H_j H_l\right]F_j\right\} \tag{7.3}$$

where

$$\mathbf{F} \equiv \mathbf{E}^* + \frac{1}{eT}(\varepsilon - \zeta)\,\mathrm{grad}_{\mathbf{r}}\,T. \tag{7.4}$$

Hence

$$J_i = -\frac{e^2}{4\pi^3\hbar^2}\int\tau\frac{\partial\varepsilon}{\partial k_i}\left\{\frac{\partial\varepsilon}{\partial k_j} - \frac{e\tau}{m^* c}\varepsilon_{ljm}\frac{\partial\varepsilon}{\partial k_l}H_m + \left(\frac{e\tau}{m^* c}\right)^2\frac{\partial\varepsilon}{\partial k_l}H_j H_l\right\}\cdot F_j\left[1+\left(\frac{e\tau}{m^* c}\right)^2 H^2\right]^{-1}\frac{\partial f_0}{\partial\varepsilon}\,d^3k. \tag{7.5}$$

The conductivity tensor, as defined in Eq. (1.1), is then given by

$$\sigma_{ij}(H) = -\frac{e^2}{4\pi^3\hbar^2}\int\frac{\partial\varepsilon}{\partial k_i}\tau\left[1+\left(\frac{e\tau}{m^* c}\right)^2 H^2\right]^{-1}\cdot \left\{\frac{\partial\varepsilon}{\partial k_j} - \frac{e\tau}{m^* c}\varepsilon_{ljm}\frac{\partial\varepsilon}{\partial k_l}H_m + \left(\frac{e\tau}{m^* c}\right)^2\frac{\partial\varepsilon}{\partial k_l}H_l H_j\right\}\frac{\partial f_0}{\partial\varepsilon}\,d^3k. \tag{7.6}$$

For a Cartesian coordinate system with $\mathbf{H}$ along the z-axis, we obtain

$$\sigma_{xx}(\mathbf{H}) = -\frac{e^2}{4\pi^3\hbar^2}\int\tau\left(\frac{\partial\varepsilon}{\partial k_x}\right)^2\left[1+\left(\frac{e\tau}{m^* c}\right)^2 H^2\right]^{-1}\frac{\partial f_0}{\partial\varepsilon}\,d^3k, \tag{7.7}$$

$$\sigma_{xy}(\mathbf{H}) = \frac{e^2}{4\pi^3\hbar^2}\int\tau\frac{e\tau H}{m^* c}\left(\frac{\partial\varepsilon}{\partial k_x}\right)^2\left[1+\left(\frac{e\tau}{m^* c}\right)^2 H^2\right]^{-1}\frac{\partial f_0}{\partial\varepsilon}\,d^3k, \tag{7.8}$$

$$\sigma_{xz}(\mathbf{H}) = 0, \tag{7.9}$$

$$\sigma_{zz}(\mathbf{H}) = -\frac{e^2}{4\pi^3\hbar^2}\int \tau\left(\frac{\partial\varepsilon}{\partial k_z}\right)^2 \frac{\partial f_0}{\partial\varepsilon}\,d^3k. \tag{7.10}$$

Part of the simplification of (7.6) has resulted from the fact that terms in the first power of $\partial\varepsilon/\partial k_i$ vanish when integrated over k-space.

It is to be noted that

$$\sigma_{zz}(\mathbf{H}) = \sigma_{zz}(0) = \sigma_{xx}(0), \qquad H = H_z. \tag{7.11}$$

This shows that the longitudinal magnetoresistance in an isotropic solid is zero.

The relationship between energy and wave number is given by

$$\varepsilon(\mathbf{k}) = (\hbar^2/2m^*)\,[k_x{}^2 + k_y{}^2 + k_z{}^2] = \hbar^2 k^2/2m^*. \tag{7.12}$$

All terms in $k_i{}^2$ in the integrands are replaced by $k^2/3$, and the volume element is $4\pi k^2\,dk$. Thus the integrals in (7.7) to (7.10) can be replaced by integrals over energy, and the results written as follows:

$$\sigma_{xx}(\mathbf{H}) = \mathscr{C}_1, \qquad \sigma_{xy}(\mathbf{H}) = -\mathscr{D}_1, \tag{7.13}$$

$$\sigma_{xz}(\mathbf{H}) = 0, \qquad \sigma_{zz}(\mathbf{H}) = \sigma_{xx}(0) \tag{7.14}$$

for which $\mathbf{H} \equiv (0, 0, H)$ and where

$$\mathscr{C}_n \equiv -\frac{2e^2(2m^*)^{1/2}}{3\pi^2\hbar^3}\int\limits_0^\infty \frac{\tau}{1+(\omega\tau)^2}\,\varepsilon^{n+1/2}\,\frac{\partial f_0}{\partial\varepsilon}\,d\varepsilon \tag{7.15}$$

$$\mathscr{D}_n \equiv -\frac{2e^2(2m^*)^{1/2}}{3\pi^2\hbar^3}\int\limits_0^\infty \frac{\omega\tau^2}{1+(\omega\tau)^2}\,\varepsilon^{n+1/2}\,\frac{\partial f_0}{\partial\varepsilon}\,d\varepsilon. \tag{7.16}$$

The cyclotron frequency is given by $\omega = eH/m^*c$, and $e > 0$ for electrons.

With the use of (1.1) and (7.1)–(7.4), the electronic contributions to the other transport coefficients can be written down by inspection. They are

$$\mathscr{M}_{xx}^{(\mathrm{el})} = \frac{1}{eT}\,[\mathscr{C}_2 - \zeta\mathscr{C}_1], \qquad \mathscr{M}_{xy}^{(\mathrm{el})} = -\frac{1}{eT}\,[\mathscr{D}_2 - \zeta\mathscr{D}_1], \tag{7.17}$$

$$\mathscr{N}_{xx}^{(\mathrm{el})} = -\frac{1}{e}\,[\mathscr{C}_2 - \zeta\mathscr{C}_1], \qquad \mathscr{N}_{xy}^{(\mathrm{el})} = \frac{1}{e}\,[\mathscr{D}_2 - \zeta\mathscr{D}_1], \tag{7.18}$$

$$\mathscr{L}^{(el)}_{xx} = -\frac{1}{e^2 T}[\mathscr{C}_3 - 2\zeta\mathscr{C}_2 + \zeta^2\mathscr{C}_1]. \tag{7.19}$$

$$\mathscr{L}^{(el)}_{xy} = \frac{1}{e^2 T}[\mathscr{D}_3 - 2\zeta\mathscr{D}_2 + \zeta^2\mathscr{D}_1]. \tag{7.20}$$

The superscripts are used to draw particular attention to the fact that the above formulae yield only the electronic contributions, that is, those due to the charge carriers. It is well known that in materials other than the high conductivity metals, a significant thermal transport occurs through the lattice. This phonon transport can contribute appreciably to the thermal conductivity of the material. It can also affect the thermoelectric quantities through the phonon-drag effect, resulting from electron-phonon coupling (see Section 29). A detailed discussion of these phenomena is beyond the scope of this work. For orientation, the reader may consult the literature — in particular, articles by Sondheimer,[59] ter Haar and Neaves,[60] and Herring *et al.*[61]

b. Phenomenological Expression for Current Density Vector

In many instances it is customary to express the current density vector in terms of components along **E**, **H**, and **E** × **H**. An expression of this sort follows directly from (7.5):

$$J_i = -\frac{e^2}{4\pi^3\hbar^2}\int \tau\left(\frac{\partial\varepsilon}{\partial k_i}\right)^2 \cdot \tag{7.21}$$

$$\left\{F_i - \left(\frac{e\tau}{m^* c}\right)(\mathbf{F}\times\mathbf{H})_i + \left(\frac{e\tau}{m^* c}\right)^2 \mathbf{F}\cdot\mathbf{H}H_i\right\}\left[1 + \left(\frac{e\tau}{m^* c}\right)^2 H^2\right]^{-1}\frac{\partial f_0}{\partial\varepsilon}\,d^3k.$$

In the above expression, the subscript *i*, although repeated, is not a summation index, but rather a component in the Cartesian coordinate system. It thus follows that for the isothermal case of our isotropic solid, we may write

$$\mathbf{J} = \sigma_H\mathbf{E}^* + \alpha_H\mathbf{E}^*\times\mathbf{H} + \gamma_H\mathbf{E}^*\cdot\mathbf{H}\mathbf{H} \tag{7.22}$$

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[59] E. H. Sondheimer, *Proc. Roy. Soc.* **A234**, 391 (1956).

[60] D. ter Haar and A. Neaves, *Advances in Phys.* **5**, 241 (1956).

[61] C. Herring, T. Geballe, and J. Kunzler, *Phys. Rev.* **111**, 36 (1958).

where

$$\sigma_H \equiv -\frac{e^2}{4\pi^3\hbar^2}\int \tau\left(\frac{\partial\varepsilon}{\partial k_i}\right)^2\left[1+\left(\frac{e\tau}{m^*c}\right)^2 H^2\right]^{-1}\frac{\partial f_0}{\partial\varepsilon}\,d^3k, \qquad (7.23)$$

$$\alpha_H \equiv \frac{e^2}{4\pi^3\hbar^2}\int \tau\left(\frac{\partial\varepsilon}{\partial k_i}\right)^2\left(\frac{e\tau}{m^*c}\right)\left[1+\left(\frac{e\tau}{m^*c}\right)^2 H^2\right]^{-1}\frac{\partial f_0}{\partial\varepsilon}\,d^3k, \qquad (7.24)$$

$$\gamma_H \equiv -\frac{e^2}{4\pi^3\hbar^2}\int \tau\left(\frac{\partial\varepsilon}{\partial k_i}\right)^2\left(\frac{e\tau}{m^*c}\right)^2\left[1+\left(\frac{e\tau}{m^*c}\right)^2 H^2\right]^{-1}\frac{\partial f_0}{\partial\varepsilon}\,d^3k \qquad (7.25)$$

where $i \equiv x, y,$ or z. Also,

$$\left(\frac{\partial\varepsilon}{\partial k_x}\right)^2 = \left(\frac{\partial\varepsilon}{\partial k_y}\right)^2 = \left(\frac{\partial\varepsilon}{\partial k_z}\right)^2 = \frac{\hbar^4}{m^{*2}}\,.$$

The factor α_H is related to the Hall coefficient, and γ_H is one of the magnetoresistance constants.

A relationship of the form (7.22) was established by Seitz.[62] His coefficients are given through second order terms in H, the development being sufficiently general so that they can be applied to solids of cubic symmetry.

To first terms in H or H^2 in (7.22), our coefficients α_H and γ_H become identical with those of Seitz, and our σ_H is related to his σ_0 and β by

$$\sigma_H = \sigma_0 + \beta H^2. \qquad (7.26)$$

8. Transport Coefficients in Anisotropic Isothermal Solids

a. Conductivity Tensors

The usual procedure in the case of anisotropic solids is to express the transport coefficients in power series of the magnetic field intensity. In this connection we note the fact that a second rank tensor is always divisible into symmetric and antisymmetric parts. From the Kohler-Onsager relation,

$$\sigma_{ij}(\mathbf{H}) = \sigma_{ji}(-\mathbf{H}), \qquad (8.1)$$

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[62] F. Seitz, *Phys. Rev.* **79**, 372 (1950). For arbitrary H, consult Garcia-Moliner.[428]

it follows that the symmetric tensor must contain only even powers of the magnetic field, and the antisymmetric tensor only odd powers.[63, 64] One may therefore write

$$\sigma_{ij}(\mathbf{H}) = \sigma_{ij}{}^0 + \alpha_{ijl} H_l + \beta_{ijlm} H_l H_m + \gamma_{ijlmn} H_l H_m H_n + \zeta_{ijlmnp} H_l H_m H_n H_p + \ldots \tag{8.2}$$

where

$$\alpha_{ijl} = \left(\frac{\partial \sigma_{ij}(\mathbf{H})}{\partial H_l}\right)_{H=0}, \tag{8.3a}$$

$$\beta_{ijlm} = \frac{1}{2!}\left(\frac{\partial^2 \sigma_{ij}(\mathbf{H})}{\partial H_l\, \partial H_m}\right)_{H=0}, \tag{8.3b}$$

$$\gamma_{ijlmn} = \frac{1}{3!}\left(\frac{\partial^3 \sigma_{ij}(\mathbf{H})}{\partial H_l\, \partial H_m\, \partial H_n}\right)_{H=0}, \tag{8.4a}$$

$$\zeta_{ijlmnp} = \frac{1}{4!}\left(\frac{\partial^4 \sigma_{ij}(\mathbf{H})}{\partial H_l\, \partial H_m\, \partial H_n\, \partial H_p}\right)_{H=0}. \tag{8.4b}$$

The validity of the expansion is the same as for Eq. (6.10), namely, that $e\tau H/m^* c \equiv \omega\tau < 1$. In Eq. (8.2), $\sigma_{ij}{}^0$ is the zero-magnetic-field conductivity; the third and fifth rank tensors are associated with the Hall conductivity, and the fourth and sixth rank tensors are associated with the magnetoconductivity.

It is often customary to use the notation in which all the above tensors are designated by σ, their galvanomagnetic association being apparent from the rank. Thus, the electric current density is written as

$$J_i = \sigma^0_{ij} E_j + \sigma^0_{ijl} E_j H_l + \sigma^0_{ijlm} E_j H_l H_m + \sigma^0_{ijlmn} E_j H_l H_m H_n + \ldots, \quad \omega\tau < 1. \tag{8.5}$$

We choose to use the superscript to differentiate these field-independent tensors from the general transport coefficients $\sigma_{ij}(\mathbf{H})$ which appear throughout this volume. It will be seen later that in cubic solids the

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[63] See p. 12 of reference 1.

[64] W. Mason, W. Hewitt, and R. Wick, *J. Appl. Phys.* **24**, 166 (1953).

σ^0_{ijl} are associated with the nondirectional Hall effect. The σ^0_{ijlmn} terms give directional contributions to the Hall effect. From the use of (6.10), (6.12), and (7.1), it is readily established that the tensors in (8.5) are given by the following expressions:

$$\sigma^0_{ij} = -\frac{e^2}{4\pi^3\hbar^2}\int \tau \frac{\partial f_0}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial k_i}\frac{\partial \varepsilon}{\partial k_j}\,d^3k, \tag{8.6}$$

$$\sigma^0_{ijl} = \frac{e^3}{4\pi^3\hbar^4 c}\int \tau \frac{\partial f_0}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial k_i}\frac{\partial \varepsilon}{\partial k_p}\frac{\partial}{\partial k_q}\left(\tau\frac{\partial \varepsilon}{\partial k_j}\right)\varepsilon_{lpq}\,d^3k, \tag{8.7}$$

$$\sigma^0_{ijlm} = -\frac{e^4}{4\pi^3\hbar^6 c^2}\int \tau \frac{\partial f_0}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial k_i}\frac{\partial \varepsilon}{\partial k_r}\frac{\partial}{\partial k_s}\left[\tau\frac{\partial \varepsilon}{\partial k_p}\frac{\partial}{\partial k_q}\left(\tau\frac{\partial \varepsilon}{\partial k_j}\right)\right]\mathscr{E}_{lrs,mpq}\,d^3k$$

$$e > 0 \text{ for electrons.} \tag{8.8}$$

In the last expression, the double permutation tensor is commonly given as the product of two standard permutation tensors, namely,

$$\mathscr{E}_{lrs,mpq} = \varepsilon_{lrs}\,\varepsilon_{mpq}. \tag{8.8a}$$

It may, however, equally well be written as

$$\mathscr{E}_{lrs,mpq} = \varepsilon_{mrs}\,\varepsilon_{lpq}. \tag{8.8b}$$

The important point is that (8.8a) and (8.8b) yield different results when (8.8) is evaluated for the expansion coefficient σ^0_{ijlm}. This is readily established by choosing a very simple case, namely, σ^0_{xyxy} and spherical energy surfaces.

The above situation does not cause errors in practice inasmuch as the summation over indices l and m causes (8.8a) and (8.8b) always to occur in pairs. Nevertheless, from a formalistic standpoint, it appears preferable to use coefficients in (8.5) which are symmetric in the indices lm. That is,

$$\sigma^0_{ijlm} = \sigma^0_{ijml}. \tag{8.9}$$

That such an arrangement is logically the most satisfactory follows from direct inspection of such relations as (6.10), (8.3b), or (8.5). A number of authors,[65, 66] for example, have discussed the symmetry of the mag-

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[65] C. Herring, *Bell System Tech. J.* **34**, 237 (1955).

[66] M. Kohler, *Ann. Physik* **20**, 891 (1934).

netoconductivity and magnetoresistivity tensors in the indices representing the components of the magnetic field, and have introduced the symmetry condition explicitly in the expression for σ^0_{ijlm}.[65] This is readily accomplished in (8.8) by expressing the double permutation tensor as follows[67]:

$$\mathscr{E}_{lrs,mpq} = \tfrac{1}{2}\,[\varepsilon_{lrs}\,\varepsilon_{mpq} + \varepsilon_{mrs}\,\varepsilon_{lpq}]. \tag{8.10}$$

Symmetric and antisymmetric relationships with interchange of the indices representing components of current and electric field also occur as a result of the Kohler-Onsager reciprocity relations. For example, if (8.1) is applied to (8.2), it is apparent from the required equality of terms that

$$\begin{gathered}\sigma^0_{ij} = \sigma^0_{ji}, \qquad \sigma^0_{ijl} = -\,\sigma^0_{jil}, \qquad \sigma^0_{iil} = 0 \quad \text{if} \quad i = j,\\ \sigma^0_{ijlm} = \sigma^0_{jilm}, \quad \text{if the coefficients are symmetric in } l \text{ and } m.\end{gathered} \tag{8.11}$$

If symmetry in l and m does not exist in the magnetoconductivity coefficients — that is, if (8.8a) or (8.8b) is used instead of (8.10) — then the latter expression in (8.11) must be replaced by

$$\sigma^0_{ijlm} = \sigma^0_{jiml}. \tag{8.11a}$$

This latter relation follows directly from Eq. (8.8), and is shown by Jones[67a] through a transformation of the integral. It also follows that σ^0_{iiii} and σ^0_{iijj} are always negative.

Much space in the literature is devoted to magnetoconductivity in crystals possessing cubic symmetry. For cubic group O_h (point group $m3m$) the different nonvanishing conductivity components through the fourth rank tensors are given by[64]

$$\begin{aligned}\sigma^0_{ij}&: \sigma^0_{11},\\ \sigma^0_{ijl}&: \sigma^0_{123},\\ \sigma^0_{ijlm}&: \sigma^0_{1111}, \sigma^0_{1122}, \sigma^0_{1212}.\end{aligned} \tag{8.12}$$

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[67] J. R. Drabble and R. Wolfe, *Proc. Phys. Soc.* (*London*) **B69**, 1101 (1956).

[67a] H. Jones, *in* "Handbuch der Physik" (S. Flügge, ed.), Vol. 19, p. 300. Springer, Berlin, 1956.

All other components of these tensors are related as follows:

$$\sigma^0_{33} = \sigma^0_{22} = \sigma^0_{11}, \qquad \sigma^0_{ij} = 0 \quad \text{if} \quad i \neq j,$$

$$\sigma^0_{ijl} = \varepsilon_{ijl}\,\sigma^0_{123}. \tag{8.12a}$$

Because of the symmetry in indices ij and lm, the fourth rank magnetoconductivity tensor can be displayed as a 6 × 6 matrix. Thus, the values of all the elements are expressed in the following matrix[64]:

$$\sigma^0_{ijlm} \equiv \sigma^0_{(ij),(lm)} = \begin{bmatrix} \sigma^0_{1,1} & \sigma^0_{1,2} & \sigma^0_{1,2} & 0 & 0 & 0 \\ \sigma^0_{1,2} & \sigma^0_{1,1} & \sigma^0_{1,2} & 0 & 0 & 0 \\ \sigma^0_{1,2} & \sigma^0_{1,2} & \sigma^0_{1,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^0_{6,6} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma^0_{6,6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma^0_{6,6} \end{bmatrix} \tag{8.12b}$$

where

$$(11) = 1; \qquad (22) = 2; \qquad (33) = 3; \tag{8.12c}$$

$$(23) = (32) = 4; \qquad (13) = (31) = 5; \qquad (12) = (21) = 6.$$

In addition to cubic group O_h (point group $m3m$) the above representation also applies to cubic systems O and T_d (point groups 432 and $\bar{4}3m$, respectively). In the case of cubic groups T and T_h (23 and $m3$, respectively), there is an additional nonvanishing coefficient, namely, $\sigma^0_{1,3}$,[67b,c] which is different from the three independent coefficients listed above.

An equation including second-order terms in magnetic field strength often used in analyzing magnetoconductivity in cubic solids is due to Seitz[62] and can be written as follows:

$$\mathbf{J} = \sigma_0\mathbf{E} + \alpha(\mathbf{E}\times\mathbf{H}) + \beta H^2\mathbf{E} + \gamma(\mathbf{E}\cdot\mathbf{H})\mathbf{H} + \delta\mathsf{T}\mathbf{E}, \qquad \omega\tau \ll 1 \tag{8.13}$$

where the coordinate axes are coincident with the crystal axes, and where T is a diagonal tensor with the elements $H_1{}^2, H_2{}^2, H_3{}^2$. In tensor notation, and taking account of the symmetry in l and m, this may be written

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[67b] See, for example, Table VIII, in C. S. Smith, in *Solid State Phys.* **6**, 175 (1958).

[67c] L. P. Kao and E. Katz, *Phys. and Chem. Solids* **6**, 223 (1958).

$$J_i = \sigma_0\,\delta_{ij}\,E_j + \varepsilon_{ijl}\,\alpha\,E_j\,H_l + \beta\delta_{ij}\,\delta_{lm}\,H_l\,H_m + \tfrac{1}{2}\gamma\,[\delta_{im}\,\delta_{lj} + \delta_{il}\,\delta_{mj}]E_j\,H_l\,H_m + \delta\delta_{ij}\,\delta_{il}\,\delta_{im}\,E_j\,H_l\,H_m \tag{8.14}$$

where summation is over all indices except i, and where the coefficient δ is not to be confused with the Kronecker deltas.

A comparison of (8.5) and (8.13) reveals that

$$\sigma^0_{ij} = \sigma_0\,\delta_{ij}, \qquad \sigma^0_{ijl} = \varepsilon_{ijl}\,\alpha, \tag{8.15}$$

$$\tfrac{1}{2}[\sigma^0_{ijlm} + \sigma^0_{ijml}] = \beta\delta_{ij}\,\delta_{lm} + \tfrac{1}{2}\gamma\,[\delta_{im}\,\delta_{lj} + \delta_{il}\,\delta_{mj}] + \delta\delta_{ij}\,\delta_{il}\,\delta_{im}. \tag{8.16}$$

The form of the expressions in brackets is chosen so that the symmetry in the indices l and m is explicitly expressed.

By using appropriate values for the indices in (8.16) one obtains the following set of relations:

$$\sigma^0_{1111} = \beta + \gamma + \delta, \qquad \sigma^0_{1122} = \beta, \qquad \sigma^0_{1212} = \gamma/2. \tag{8.17}$$

If (8.15) and (8.16) are evaluated for all sets of indices, one obtains again all the information presented in (8.12a) and (8.12b).

From (8.17) we see that the Seitz coefficients are related to the basic conductivity components of (8.12) as follows:

$$\sigma_0 = \sigma^0_{11}, \qquad \alpha = \sigma^0_{123}, \qquad \beta = \sigma^0_{1122},$$
$$\gamma = 2\sigma^0_{1212}, \qquad \delta = \sigma^0_{1111} - \sigma^0_{1122} - 2\sigma^0_{1212}. \tag{8.18}$$

In the case of isotropic materials,

$$\delta = 0, \qquad \beta + \gamma = 0. \tag{8.18a}$$

b. Resistivity Tensors

In many galvanomagnetic measurements, the current density is maintained constant, and the electric field, which is permitted to adjust itself, is determined from measurements of potential differences.

In such cases, it is desirable to express electrical transport effects in terms of the resistivity tensor $\boldsymbol{\rho}(\mathbf{H})$, as is done in (2.4). For the isothermal case,[68]

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[68] For homogeneous materials with zero temperature gradients, it is to be noted that $\mathbf{E}^* = \mathbf{E}$. Since many of the sections to follow are concerned with such cases, the quantity $\mathbf{E}$ will appear in most equations.

$$E_i = \rho_{ik}(\mathbf{H}) J_k. \tag{8.19}$$

The above expression is often written in a way analogous to (8.5), namely,

$$E_i = \rho^0_{ik} J_k + \rho^0_{ikl} J_k H_l + \rho^0_{iklm} J_k H_l H_m + \rho^0_{iklmn} J_k H_l H_m H_n + \dots, \qquad \omega\tau < 1. \tag{8.20}$$

The higher rank tensors ρ^0_{ikl}, ρ^0_{iklm}, ... are commonly known as galvanomagnetic coefficients. They can be expressed explicitly in terms of the conductivity coefficients of (8.5) through the use of relation (2.6), namely, $\rho_{ij}(\mathbf{H})\sigma_{jk}(\mathbf{H}) = \delta_{ik}$.

Another form of (8.19), given by Casimer,[69,70] can be written as follows:

$$E_i = \rho^s_{ik}(\mathbf{H}) J_k + [\mathbf{R}(\mathbf{H}) \times \mathbf{J}]_i. \tag{8.21}$$

The sign convention used above is such that the first term in the $\mathbf{R}(\mathbf{H})$ expansion for the isotropic case is $R\mathbf{H}$, where R is precisely the ordinary Hall coefficient. In (8.21) the resistivity tensor has been split into a symmetrical part $\rho^s_{ik}(\mathbf{H})$, which is an even function of the magnetic field, and an antisymmetrical part $\rho^a_{ik}(\mathbf{H})$ expressed in terms of a *Hall vector*[71] $\mathbf{R}(\mathbf{H})$. The Hall vector is an odd function of $\mathbf{H}$, with components

$$R_1(\mathbf{H}) = \rho^a_{32}(\mathbf{H}), \qquad R_2(\mathbf{H}) = \rho^a_{13}(\mathbf{H}), \qquad R_3(\mathbf{H}) = \rho^a_{21}(\mathbf{H}). \tag{8.23}$$

In tensor notation, (8.23) and (8.21) may be written

$$R_s(\mathbf{H}) = \tfrac{1}{2} \varepsilon_{sut}\, \rho^a_{tu}(H), \tag{8.24}$$

$$E_i = \rho^s_{ik}(\mathbf{H}) J_k + \varepsilon_{isk}\, R_s(\mathbf{H}) J_k. \tag{8.25}$$

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[69] H. B. G. Casimer and A. N. Gerritsen, *Physica* **8**, 1107 (1941).

[70] H. B. G. Casimer, *Revs. Modern Phys.* **17**, **343** (1945).

[71] With $\mathbf{R}(\mathbf{H})$ defined as a Hall vector and the last term of (8.21) therefore representing the Hall field, it follows that the Hall effect thus defined is an odd function of magnetic field. This definition, although frequently adhered to, is not universally followed. Some authors adopt a more general definition, which permits the occurrence of even powers of H in the Hall terms. These considerations are discussed in Chapter IV.

Since $R_s(\mathbf{H})$ is odd in H, its general expansion is of the form

$$R_s(\mathbf{H}) = R^0_{sl} H_l + R^0_{slmn} H_l H_m H_n + \ldots, \qquad \omega\tau < 1. \tag{8.26}$$

It is thus apparent that (8.25) and (8.26) are equivalent to (8.20) provided that

$$R^0_{sl} = \tfrac{1}{2} \varepsilon_{sut} \rho^0_{tul}, \qquad R^0_{slmn} = \tfrac{1}{2} \varepsilon_{sut} \rho^0_{tulmn} \tag{8.27}$$

or that

$$\varepsilon_{isk} R^0_{sl} = \rho^0_{ikl}, \qquad \varepsilon_{isk} R^0_{slmn} = \rho^0_{iklmn}. \tag{8.28}$$

There is also the obvious relation that

$$\rho^s_{ik}(\mathbf{H}) = \rho^0_{ik} + \rho^0_{iklm} H_l H_m + \ldots, \qquad \omega\tau < 1.$$

The factor of $\frac{1}{2}$ appears in expressions (8.27) because the summations there are over the two indices in which the tensors are antisymmetric.

c. *Application of Phenomenological Relations to Various Crystal Systems*

(*1*) *General systems.* The number of nonvanishing independent components of a given order galvanomagnetic coefficient is determined by physical requirements such as expressed in (8.9) and (8.11) and by the symmetry of the material. The application of group theory and direct inspection procedures has been applied by a number of investigators[67b,c,72–78c] to establish these independent components and to determine the other components of the tensors. Results are available for all of the crystallographic groups. Terms as high as the sixth order have been considered.

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[72] M. Kohler, *Ann. Physik* **20**, 878, 891 (1934).

[73] D. Shoenberg, *Proc. Cambridge Phil. Soc.* **31**, 265 (1935).

[74] H. J. Juretschke, *Acta Cryst.* **5**, 148 (1952).

[75] F. G. Fumi, *Acta Cryst.* **5**, 44, 691 (1952).

[76] F. G. Fumi, *Nuovo cimento* **9**, 739 (1952).

[77] R. Fieschi and F. G. Fumi, *Nuovo cimento* **10**, 865 (1953).

[78] H. B. Huntington, *Solid State Phys.* **7**, 213 (1958).

[78a] H. Bross, *Z. Naturforsch.* **15a**, 859 (1960).

[78b] G. F. Koster, *Solid State Phys.* **5**, 173 (1957).

[78c] A. V. Sokolov and V. P. Shirokovskii, *Uspekhi Fiz. Nauk* **71**, 485 (1960) [translation: *Soviet Phys.-Uspekhi* **3**, 551 (1961)].

(2) *Cubic groups O_h, O, or T_d (systems of point group symmetry* **m3m**, *432, 43m*). The five different nonvanishing conductivity components through the terms involving second powers of the magnetic field, and their relation to the constants in the equation of Seitz, have been given in Subsection *a*. These conductivity coefficients may be transformed to the galvanomagnetic coefficients in Eq. (8.20) by use of the reciprocal relation

$$\rho_{ij}(\mathbf{H})\sigma_{jp}(\mathbf{H}) = \delta_{ip}. \tag{8.29}$$

In using this relation, care must be taken to take the interchangeability of the magnetic field indices into account. This is readily done by writing a symmetric form whenever two or more such indices occur. This procedure is outlined by the several steps below:

$$\rho_{ij}(H) = \rho^0_{ij} + \rho^0_{ijl} H_l + \tfrac{1}{2}(\rho^0_{ijlm} + \rho^0_{ijml}) H_l H_m + \ldots,$$

$$\sigma_{jp}(H) = \sigma^0_{jp} + \sigma^0_{jpm} H_m + \tfrac{1}{2}(\sigma^0_{jplm} + \sigma^0_{jpml}) H_l H_m + \ldots. \tag{8.29a}$$

Carrying out summations over j yields, for the product (8.29),

H^0 terms: $\rho^0_{i1}\sigma^0_{1p} + \rho^0_{i2}\sigma^0_{2p} + \rho^0_{i3}\sigma^0_{3p} = \delta_{ip}.$

H terms: $\rho^0_{i1}\sigma^0_{1pl} + \rho^0_{i2}\sigma^0_{2pl} + \rho^0_{i3}\sigma^0_{3pl} + \sigma^0_{1p}\rho^0_{i1l} + \sigma^0_{2p}\rho^0_{i2l} + \sigma^0_{3p}\rho^0_{i3l} = 0.$
$(m = l)$

H^2 terms:
$$\begin{aligned}&\tfrac{1}{2}(\rho^0_{i1l}\sigma^0_{1pm} + \rho^0_{i1m}\sigma^0_{1pl}) + \tfrac{1}{2}(\rho^0_{i2l}\sigma^0_{2pm} +\\ &\quad \rho^0_{i2m}\sigma^0_{2pl}) + \tfrac{1}{2}(\rho^0_{i3l}\sigma^0_{3pm} + \rho^0_{i3m}\sigma^0_{3pl}) + \tfrac{1}{2}\rho_{i1}(\sigma^0_{1plm} + \sigma^0_{1pml}) +\\ &\tfrac{1}{2}\rho_{i2}(\sigma^0_{2plm} + \sigma^0_{2pml}) + \tfrac{1}{2}\rho_{i3}(\sigma^0_{3plm} + \sigma^0_{3pml}) + \tfrac{1}{2}\sigma^0_{1p}(\rho^0_{i1lm} + \rho^0_{i1ml}) +\\ &\tfrac{1}{2}\sigma^0_{2p}(\rho^0_{i2lm} + \rho^0_{i2ml}) + \tfrac{1}{2}\sigma^0_{3p}(\rho^0_{i3lm} + \rho^0_{i3ml}) = 0.\end{aligned} \tag{8.29b}$$

The above relationships are completely general. For the basic components of the cubic system, given in (8.12) they yield

$$\sigma^0_{11} = 1/\rho^0_{11}, \quad \sigma^0_{123} = -\rho^0_{123}/\rho^{0\,2}_{11}, \quad \sigma^0_{1111} = -\rho^0_{1111}/\rho^{0\,2}_{11},$$

$$\sigma^0_{1122} = -\rho^0_{1122}/\rho^{0\,2}_{11} - \rho^{0\,2}_{123}/\rho^{0\,3}_{11}, \quad \sigma^0_{1212} = -\rho^0_{1212}/\rho^{0\,2}_{11} + \tfrac{1}{2}\rho^{0\,2}_{123}/\rho^{0\,3}_{11}. \tag{8.29c}$$

An inverted form of (8.13), again to second order in H, due to Pearson and Suhl[79] may be written

$$\mathbf{E} = \rho_0[\mathbf{J} + a\mathbf{J} \times \mathbf{H} + bH^2\,\mathbf{J} + c(\mathbf{J} \cdot \mathbf{H})\mathbf{H} + d\mathbf{T}\,\mathbf{J}], \qquad \omega\tau < 1. \tag{8.30}$$

The constants in the above expression are related, to within H^2 terms, to those in (8.13) as follows:

$$\begin{aligned} \rho_0 &= 1/\sigma_0 & c &= -(\gamma - \alpha^2/\sigma_0)\sigma_0 \\ a &= -\alpha/\sigma_0 & d &= -\delta/\sigma_0, \\ b &= -(\beta + \alpha^2/\sigma_0)/\sigma_0 \end{aligned} \tag{8.31}$$

Some authors write the weak-field Hall coefficient, in place of the factor $a\rho_0$, although others are inclined to interpret the Hall effect in a more general manner.[71] In our notation this would give

$$a\rho_0 = -R. \tag{8.31a}$$

The relationships between the constants in (8.30) and the galvanomagnetic coefficients are similar to those given in (8.18), namely,

$$\rho_0 = \rho^0_{11}, \quad a = \rho^0_{123}/\rho_0, \quad b = \rho^0_{1122}/\rho_0, \tag{8.31b}$$
$$c = 2\rho^0_{1212}/\rho_0, \quad d = [\rho^0_{1111} - \rho^0_{1122} - 2\rho^0_{1212}]/\rho_0.$$

In the case of isotropic materials,

$$d = 0, \qquad b + c = 0. \tag{8.31c}$$

Higher order terms in H, for Hall effect and for magnetoresistance, have been taken into account by Mason *et al.*[64] The effect of these terms can be appreciable when the magnetic field is not along a crystallographic axis of symmetry. It turns out that there are two independent constants necessary to describe the second-order terms of the Hall effect (i.e., the terms of the third power in H); and that six constants are required for a complete description of the fourth power terms in H.

Considerable data on the determination of the galvanomagnetic coefficients of germanium and silicon, including contributions by the authors quoted in this section, are available. These will be discussed in the sections dealing specifically with those semiconductors.

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[79] G. L. Pearson and H. Suhl, *Phys. Rev.* **83**, 768 (1951).

(3) *System of point group symmetry* $3m$ *or* $\bar{3}m$ (*trigonal system* C_{3v} or D_{3d}). In this case there are a total of 12 different nonzero components of the tensors $\rho^0_{ik}, \rho^0_{ikl}, \rho^0_{iklm}$, each tensor requiring respectively 2, 2, and 8 different constants. A detailed study of the coefficients including those for terms up to the fourth power in H has been presented by Juretschke.[80] The third-power terms involve 6 constants, while those of the fourth power in H require 18 constants for their specification. The 12 independent lower order terms and their relation to the corresponding conductivity coefficients are shown below for a Cartesian coordinate system with direction 3 parallel to the threefold axis of the crystal and with the (2,3) plane as one of the three equivalent planes of reflection, i.e., coordinate direction 1 is along one of the three twofold (binary) axes in the case of $\bar{3}m$ systems[81]:

$$\rho^0_{11} = 1/\sigma^0_{11}, \qquad \rho^0_{33} = 1/\sigma^0_{33}; \qquad \rho^0_{123} = -\,\sigma^0_{123}/\sigma^{0\,2}_{11}, \qquad \rho^0_{231} = -\,\sigma^0_{231}/\sigma^0_{11}\,\sigma^0_{33}, \tag{8.32}$$

$$\begin{aligned}
\rho^0_{1111} &= -\,\sigma^0_{1111}/\sigma^{0\,2}_{11}; & \rho^0_{1122} &= -\,\sigma^0_{1122}/\sigma^{0\,2}_{11} - \sigma^{0\,2}_{231}/\sigma^{0\,2}_{11}\,\sigma^0_{33} \\
\rho^0_{3333} &= -\,\sigma^0_{3333}/\sigma^{0\,2}_{33}; & \sigma^0_{1133} &= -\,\sigma^0_{1133}/\sigma^{0\,2}_{11} - \sigma^{0\,2}_{123}/\sigma^{0\,3}_{11}, \\
\rho^0_{1123} &= -\,\sigma^0_{1123}/\sigma^{0\,2}_{11}; & \rho^0_{3311} &= -\,\sigma^0_{3311}/\sigma^{0\,2}_{33} - \sigma^{0\,2}_{231}/\sigma^0_{11}\,\sigma^{0\,2}_{33}, \\
\rho^0_{2311} &= -\,\sigma^0_{2311}/\sigma^0_{11}\,\sigma^0_{33}; & \rho_{2323} &= -\,\sigma^0_{2323}/\sigma^0_{11}\,\sigma^0_{33} + \tfrac{1}{2}\,\sigma^0_{123}\,\sigma^0_{231}/\sigma^{0\,2}_{11}\,\sigma^0_{33}.
\end{aligned} \tag{8.33}$$

The above equalities hold if ρ^0 and σ^0 are interchanged throughout.

The other components of the resistivity and Hall effect tensors, not given in (8.32), are as follows:

$$\rho^0_{22} = \rho^0_{11}, \qquad \rho^0_{ik} = 0 \qquad \text{if} \qquad i \neq k, \tag{8.34}$$

$$\rho^0_{312} = \rho^0_{231}, \qquad \rho^0_{ikl} = 0 \qquad \text{if} \quad i = k \text{ or } k = l \text{ or } l = i. \tag{8.35}$$

All the components of the magnetoresistance tensor are displayed in the matrix below:

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[80] H. J. Juretschke, *Acta Cryst.* 8, 716 (1955).

[81] The statement in Juretschke's[80] article that the plane of (x, z) is one of the three equivalent planes of reflection should be corrected to read the plane of (y, z) [Juretschke, private communication]. The expressions applicable when the (x, z) coordinate plane is a plane of reflection have been derived by Drabble, Groves, and Wolfe. See p. 442 in reference 87.

$$\rho^0_{(ik),(lm)} = \begin{bmatrix} \rho^0_{1,1} & \rho^0_{1,2} & \rho^0_{1,3} & \rho^0_{1,4} & 0 & 0 \\ \rho^0_{1,2} & \rho^0_{1,1} & \rho^0_{1,3} & -\rho^0_{1,4} & 0 & 0 \\ \rho^0_{3,1} & \rho^0_{3,1} & \rho^0_{3,3} & 0 & 0 & 0 \\ \rho^0_{4,1} & -\rho^0_{4,1} & 0 & \rho^0_{4,4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho^0_{4,4} & \rho^0_{4,1} \\ 0 & 0 & 0 & 0 & \rho^0_{1,4} & \frac{1}{2}(\rho^0_{1,1} - \rho^0_{1,2}) \end{bmatrix} \tag{8.36}$$

where, as in (8.12c),

$$(11) = 1; \qquad (22) = 2; \qquad (33) = 3;$$
$$(23) = (32) = 4; \qquad (13) = (31) = 5; \qquad (12) = (21) = 6.$$

The above relations have been used by Freedman and Juretschke[82] in an investigation of antimony. All 12 independent components of the resistivity tensor through second power terms in H were measured. A theoretical treatment was then carried out using three-valleyed bands for both valence and conduction bands and energy-independent isotropic relaxation times. The theory contains 9 parameters: three principal mobilities for electrons and three for holes, an angle of tilt of one of the principal axes of the electron energy ellipsoid out of the base plane, a corresponding angle for holes, and the carrier density. The best values of these parameters to fit the experimental data were obtained with the aid of an IBM 650 computer.

The preceding relationships have also been applied to investigations of the galvanomagnetic coefficients of bismuth and of bismuth telluride. Substantial experimental data on bismuth are presented by Okada.[83,84] Theoretical calculations were made by Abeles and Meiboom[85] by approximating the band structure by a many-valley model in which the energy surfaces are approximated by ellipsoids. These are arranged in momentum

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[82] S. J. Freedman and H. J. Juretschke, Tech. Rept. No. 6, Contract NONR 839(06) (April 15, 1959). Subject report forms part of a Ph.D. thesis presented by S. J. Freedman in the Physics Department of the Polytechnic Institute of Brooklyn. Most of the work has subsequently appeared in print: S. J. Freedman and H. M. Juretschke, *Phys. Rev.* **124**, 1379 (1961).

[83] T. Okada, *J. Phys. Soc. Japan* **11**, 89 (1956).

[84] T. Okada, *J. Phys. Soc. Japan* **12**, 1327 (1957).

[85] B. Abeles and S. Meiboom, *Phys. Rev.* **101**, 544 (1956).

space in configurations appropriate for the symmetry of the crystal. A constant relaxation time is assumed. Low-temperature galvanomagnetic data at intermediate field strengths are also available in a number of publications.[65a, b]

A similar model was used by Coldwell-Horsfall and ter Haar.[66] They, however, choose a relaxation time proportional to $\varepsilon^{-1/2}$, i.e., a constant mean free path. These authors are careful to use the three-dimensional transport equations in calculating the Hall effect. The point is made that, although in the isotropic case with **J** normal to **H** it follows that there is no component of **E** along **H**, the statement is not generally true in the case of anisotropy. For bismuth, with **H** along the binary axis, it is calculated that E in the direction of **H** might amount to the order of 0.1% $|E|$, and should be measurable.

A similar procedure to that used for bismuth was adopted by Drabble and Wolfe for bismuth telluride.[67] Both the conduction[67] and valence[67a] bands are considered, and these authors modify Eqs. (8.32) – (8.36) to provide expressions which are applicable when one of the three equivalent reflection planes of the crystal is coincident with the (x, z) coordinate plane.[81]

Further information on the many-valley model and its application to a number of semiconductors, including a discussion of the results for bismuth telluride, is given in Chapter VIII.

(*4*) *Orthorhombic system* (*group* D_{2h}). For this system, the number of independent components of the resistivity tensors are 3 for the zero-magnetic-field resistivity, 3 for the first power Hall term, 12 for the magnetoresistance terms, and 9 for the H^3 Hall term.[67c]

The lower order galvanomagnetic coefficients are discussed in Kohler's article.[72] However, the number of independent relations for the linear Hall term — namely, 6 as determined from the crystal symmetry — is

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[65a] R. A. Connell and J. A. Marcus, *Phys. Rev.* **107**, 940 (1957).

[65b] S. Mase and S. Tanuma, *Sci. Repts. Research Insts. Tohoku Univ.* **12**, 35 (1960).

[66] R. Coldwell-Horsfall and D. ter Haar, *Physica* **24**, 848 (1958).

[67] J. R. Drabble, R. D. Groves and R. Wolfe, *Proc. Phys. Soc.* (*London*) **71**, 430 (1958).

[67a] J. R. Drabble, *Proc. Phys. Soc.* (*London*) **72**, 380 (1958).

halved as a result of the antisymmetry in electric field and current indices. The Kohler-Onsager relations (2.10) had not yet been established in 1934 when the article in question was written.

Extensive data on a number of gallium crystals are available from measurements by Yahia and Marcus[88] at several temperatures. At 290°K and at 77°K, the galvanomagnetic properties are adequately described by the phenomenological coefficients discussed. At 4°K, however, results indicate an apparent reduction in the crystal symmetry, so that a larger number of independent components are necessary to specify the resistivity tensor, $\boldsymbol{\rho}(\mathbf{H})$. It is suggested that this might result from a sharp structure of the Fermi surface existing at low temperatures. In cases where high mobilities are encountered at low temperatures, it was noted by Jain that additional components in the resistivity tensor can become noticeable as a result of the weak-magnetic-field limit being exceeded.[88a]

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[88] J. Yahia and J. A. Marcus, *Phys. Rev.* **113**, 137 (1959).

[88a] A. L. Jain, *Phys. Rev.* **114**, 1518 (1959).

IV.

Experimental Determination of Transport Coefficients in Isothermal Media

9. MEASUREMENTS ON ISOTROPIC MATERIALS

When the anisotropy of the crystal can be neglected, the transport coefficients are specified if we know the conductivity (or resistivity) and the Hall coefficient as functions of magnetic field. The resistivity is determined with the magnetic field normal to the current — leading to transverse magnetoresistance — and also with **H** parallel to **J** — leading to longitudinal magnetoresistance. Actually, the longitudinal magnetoresistance should vanish if complete isotropy occurs. Thus, the relative smallness of this quantity is an indication as to how well the idealized conditions are realized.

In order to illustrate the contributions of the coefficients discussed above, we shall derive a vector equation for **J**(**E**). Let us consider a specimen which may deviate only slightly from isotropy, so that there is a small but nonzero longitudinal magnetoresistance yet the theory for isotropic systems can be used. The procedure is to express the conductivity coefficients of (7.22) in terms of $\sigma_\perp(H)$, the conductivity in a transverse magnetic field and $\sigma_\parallel(H)$, the conductivity in a longitudinal magnetic field.

Thus,

$$\sigma_\perp(H) = (J/E)_{\mathbf{E}\perp\mathbf{H}} = \sigma_H, \tag{9.1}$$

$$\sigma_\parallel(H) = (J/E)_{\mathbf{E}\parallel\mathbf{H}} = \sigma_H + \gamma_H H^2,$$

and Eq. (7.22) assumes the form

$$\mathbf{J} = \sigma_\perp(H)\mathbf{E} + \alpha_H\, \mathbf{E}\times\mathbf{H} + \frac{\sigma_\parallel(H) - \sigma_\perp(H)}{H^2}\, \mathbf{E}\cdot\mathbf{H}\mathbf{H}. \tag{9.1a}$$

The coefficient $\alpha_H H$ is sometimes referred to as the Hall conductivity, i.e.,

$$\alpha_H H \equiv \sigma^H(H), \tag{9.1b}$$

and with the customary choice of coordinate system

$$\sigma^H(H) \equiv \sigma_{xy}(H). \tag{9.1c}$$

In the case of exact isotropy, $\sigma_{\parallel}(H) = \sigma_{\parallel}(0) = \sigma_0$.

The inverted form of (7.22) can be written as follows [cf. (8.30)]:

$$\mathbf{E} = \rho_H \mathbf{J} - R(H)\mathbf{J} \times \mathbf{H} + C(H)\mathbf{J} \cdot \mathbf{HH}. \tag{9.1d}$$

In terms of resistivities in transverse and longitudinal magnetic fields, one has

$$\mathbf{E} = \rho_{\perp}(H)\mathbf{J} + R(H)\mathbf{H} \times \mathbf{J} + \frac{\rho_{\parallel} - \rho_{\perp}}{H^2}\mathbf{J} \cdot \mathbf{HH}. \tag{9.1e}$$

The term $R(H)$ is the Hall coefficient, and $R(H)H$ is the Hall resistivity;

$$\begin{aligned} R(H)H &= \rho^H(H), \\ &= \rho_{yx}(H) \qquad \text{if} \qquad \mathbf{H} \equiv (0, 0, H). \end{aligned} \tag{9.1f}$$

The latter equality applies for the customary choice of coordinate axes.

The above equations are similar to those given by Jan.[88b] They are mathematically exact in H, and are not expressions derived from expansions valid to certain powers in H.

The relations between the conductivities and resistivities are as follows:

$$\sigma_{\perp}(H) = \frac{\rho_{\perp}(H)}{\rho_{\perp}^2(H) + [R(H)H]^2}, \qquad \sigma_{\parallel}(H) = \frac{1}{\rho_{\parallel}(H)}, \tag{9.1g}$$

$$\alpha_H = \frac{R(H)}{\rho_{\perp}^2(H) + [R(H)H]^2}.$$

The transverse relations include contributions from the Hall effect. In determining the Hall coefficient, the common procedure is for **J** and **H**

[88b] See pp. 15–17 of reference 1. The meaning of our $\sigma_{\perp}$ is identical with that of Jan's $\gamma_{\perp}$.

to be normal to each other, and the Hall field is then measured in a direction perpendicular to each. Variations of these phenomena, which include magnetoconductivity, Corbino magnetoresistance, and planar Hall effect, will be discussed later.

a. Discussion of Transverse Magnetoresistance and Hall Effects

Consider $\mathbf{H} = (0, 0, H)$ and the current $\mathbf{J}$ to exist in the x-direction in the isothermal solid. Then relations (1.1) and (2.4) become

$$J_x = \sigma_{xx}(H)E_x + \sigma_{xy}(H)E_y + \sigma_{xz}(H)E_z, \tag{9.2}$$

$$0 = \sigma_{yx}(H)E_x + \sigma_{yy}(H)E_y + \sigma_{yz}(H)E_z, \qquad H \equiv H_z, \tag{9.3}$$

$$0 = \sigma_{zx}(H)E_x + \sigma_{zy}(H)E_y + \sigma_{zz}(H)E_z \tag{9.4}$$

and

$$E_x = \rho_{xx}(H)J_x, \qquad E_y = \rho_{yx}(H)J_x, \qquad E_z = \rho_{zx}(H)J_x. \tag{9.4a}$$

For the isotropic case, relation (3.2) requires that E_z vanish.[89] Hence, using the symmetry expressed by (3.1), we may write

$$\begin{gathered} J_x = \sigma_{xx}(H)E_x + \sigma_{xy}(H)E_y, \\ 0 = -\sigma_{xy}(H)E_x + \sigma_{xx}(H)E_y, \qquad J \equiv J_x \end{gathered} \tag{9.5}$$

and

$$E_x = \rho_{xx}(H)J_x, \qquad E_y = \rho_{yx}(H)J_x. \tag{9.6}$$

In terms of the fields and current shown above, the Hall coefficient, the conductivity, and the magnetoresistance are given by

$$R_H = E_y/J_x H, \tag{9.7}$$

$$\sigma(H) = J_x/E_x, \tag{9.8}$$

$$\Delta\rho/\rho_0 = [\rho(H) - \rho(0)]/\rho_0 = [E_x(H)/E_x(0)] - 1, \qquad J_x = \text{const.} \tag{9.9}$$

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[89] It is important to note that according to (9.2)–(9.4), E_z will vanish if and only if $\sigma_{yx}(H)\sigma_{zy}(H) - \sigma_{zx}(H)\sigma_{yy}(H) = 0$. Therefore the condition $J_y = J_z = 0$ is not sufficient to allow use of the two-dimensional relations (9.5) in the case of anisotropic media. This point, which has led to difficulties in the literature, is emphasized by Coldwell-Horsfall and ter Haar.[88]

In all of the above definitions, the boundary conditions are

$$J_y = \partial T/\partial x = \partial T/\partial y = 0, \qquad \mathbf{H} = (0, 0, H) \tag{9.10}$$

along with the obvious fact, alluded to in connection with (9.5), that

$$E_z = J_z = \partial T/\partial z = 0. \tag{9.11}$$

The requirement that $\partial T/\partial y$ vanish is imposed by our consideration of isothermal phenomena. It is possible, of course, to define adiabatic phenomena.[90] These are discussed briefly in Section 29. Then the condition that $\partial T/\partial y$ vanish is usually replaced by the requirement that the transverse heat current q_y be zero. A great simplification results in the transport equations, however, whenever the isothermal approximation can be used. As will be shown later, such a treatment is fairly good for a number of semiconductors, even when measurements are taken under conditions which are favorable to the adiabatic situation.

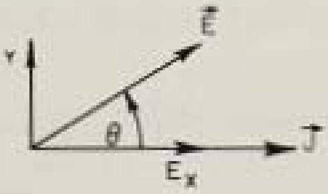

FIG. 3. Illustration of Hall angle.

From (9.5) and (9.6) it follows that for isotropic systems with $\mathbf{H} = (0, 0, H)$

$$R_H = \frac{\rho_{yx}(H)}{H} = \frac{\sigma_{xy}(H)}{H[\sigma_{xx}^2(H) + \sigma_{xy}^2(H)]}, \tag{9.12}$$

$$\sigma(H) = \frac{1}{\rho_{xx}(H)} = \frac{\sigma_{xx}^2(H) + \sigma_{xy}^2(H)}{\sigma_{xx}(H)}, \tag{9.13}$$

$$\Delta\rho/\rho_0 = \left[\frac{\rho_{xx}(H)}{\rho_{xx}(0)}\right] - 1 = \frac{\sigma_{xx}(H)\sigma_{xx}(0)}{\sigma_{xx}^2(H) + \sigma_{xy}^2(H)} - 1. \tag{9.14}$$

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[90] See p. 6 of reference 1.

Since the Hall phenomenon is essentially a rotation of the electric field vector by the magnetic field, many authors have introduced a Hall angle[91] (Fig. 3), defined as

$$\tan\theta = E_y/E_x, \qquad \mathbf{H} = (0, 0, H). \tag{9.15}$$

Although the Hall angle is most frequently employed in the weak-magnetic-field region, we shall present general equations below, which are valid for arbitrary values of H. These relations are

$$\tan\theta = \rho_{yx}(H)/\rho_{xx}(H) = \sigma_{xy}(H)/\sigma_{xx}(H), \qquad \mathbf{H} = (0, 0, H). \tag{9.16}$$

From (9.12) and (9.13) we see that

$$\tan\theta = R_H\,\sigma(H)H. \tag{9.17}$$

Further relations involving the Hall angle are discussed in Section 12*d*.

b. Experimental Determinations of Hall Coefficient and Magnetoresistance

We shall omit a discussion of measurement techniques used in the laboratory. Details can be found in a number of textbooks and articles, several of which are referenced here.[79,92–95] The specimens are usually cut in the form of rectangular parallelepipeds or in the bridge shape.[94,96] The latter design has the advantage of providing large contact areas with a reasonably small disturbance of the electric fields to be measured.

When, however, magnetoresistance is to be determined in high-mobility materials (that is, where the Hall angle is relatively large), the bridge samples with lateral arms may not be satisfactory. Even such a small amount of disturbance of the Hall field cannot be tolerated. In

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[91] See, for example, p. 209 of reference 32; p. 16 of reference 1.

[92] I. Estermann and A. Foner, *Phys. Rev.* **79**, 365 (1950).

[93] E. H. Putley, "The Hall Effect and Related Phenomena," Butterworths, London, 1960.

[93a] Reference 1, pp. 17–25.

[94] W. C. Dunlap, "An Introduction to Semiconductors," pp. 178–194. Wiley, New York, 1957.

[95] O. Lindberg, *Proc. I. R. E.* **40**, 1414 (1952).

[96] P. P. Debye and E. M. Conwell, *Phys. Rev.* **93**, 693 (1954).

those cases it is desirable to use only carefully fashioned point contacts for the potential probes. This problem is discussed in more detail later.

In order to take advantage of the simplifications of rectangular geometry it is essential that the current lines be parallel and that the electric field which is being measured to determine the Hall effect be normal to these lines. Misalignment in the Hall potential probes will add an "IR drop" to the transverse voltage. Since this spurious voltage at weak magnetic fields is independent of H but proportional to J_x, it is usually eliminated by averaging data for opposite directions of current and of magnetic field. Many problems are associated with satisfying the boundary conditions (9.10). These are discussed in subsequent paragraphs in this section.

It is also possible to determine certain galvanomagnetic coefficients on samples of various other shapes, for example, those of circular symmetry,[96a] of infinite planes,[97] and of other shapes,[97a] including arbitrary two-dimensional geometry.[98,99] By use of a "clover-shaped" sample, the influence of the contacts can be reduced considerably.[99] Such a design has several advantages over the bridge shape, such as improved heat dissipation and mechanical strength. By cutting discs with planes normal to the appropriate directions in the crystal, one can determine the directional resistivities of anisotropic conductors, as was shown by Hornstra and van der Pauw.[99a] The disk technique has subsequently been extended by van der Pauw[99b] to provide means for determining all the components of the zero-magnetic-field resistivity tensor and also those of the weak-field Hall tensor in an anisotropic solid.[99b] In the most general case, where the solid has no elements of symmetry, the six independent resistivity constants are obtained from the sheet resistivites, at zero magnetic field, of six plane-parallel samples through six linear equations. The nine constants associated with the Hall effect are established through measurements on three samples, combined with three different orienta-

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[97] J. La Plume, *L'Onde Elec.* **35**, 113 (1955); M. Wintenberger, *Compt. rend. acad. sci.* (*Paris*) **246**, 2366 (1958).

[97a] R. Jaggi and R. Sommerhalder, *Helv. Phys. Acta* **32**, 167 (1959).

[98] R. M. Broudy, *J. Appl. Phys.* **29**, 853 (1958).

[99] L. J. van der Pauw, *Philips Research Repts.* **13**, 1 (1958).

[99a] J. Hornstra and L. J. van der Pauw, *J. Electronics and Control* **7**, 169 (1959).

[99b] L. J. van der Pauw, *Philips Research Repts.* **16**, 187 (1961).

tions of the magnetic field. The measurement of weak-field magnetoresistance coefficients of cubic materials, using two flat samples of arbitrary shape, is discussed by Matthews and Doherty.[99c]

For determining resistivities alone, the four-point probe techniques, discussed by Valdes[99d] and by Uhlir,[99e] are sometimes employed. Application to anisotropic media has been discussed by Airapetyants and Bresler.[99f]

(*f*) *Effect of transverse current on Hall voltage.* It is apparent that a shorting of the Hall field by any means such as area contacts, inhomogeneous sections in the material, or external circuitry will reduce the Hall voltage. When area contacts are used at the ends, the length of the specimen must be substantial compared to the width (the magnetic field vector being along the *thickness* of the parallelepiped) so that the measured Hall voltage is not reduced because of the shorting at the ends. Another possibility would be to use point contacts at the ends. Such an arrangement however will produce nonparallel current lines near the ends, and the specimen will still need to be long in order to assume parallel current lines in the neighborhood of the Hall probes. The effects of end-contact shorting where the Hall angle is small have been studied by Isenberg *et. al.*,[100] Volger,[101] and others. The relative decrease in the measured Hall voltage is shown in Fig. 4.

Volger also uses his results to examine aspects of certain macroscopic inhomogeneities in the specimens. Analysis of geometries other than rectangular was made by Frank[102] who also examined the effects of pin contacts for current electrodes. The case of a nonuniform magnetic field was included in the investigations of Flanagan *et al.*[103] It is interesting to note that for a flat-topped bell-shaped magnetic field variation in the

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[99c] H. Matthews and W. R. Doherty, *J. Electronics and Control* **10**, 273 (1961).

[99d] L. B. Valdes, *Proc. I. R. E.* **42**, 420 (1954).

[99e] A. Uhlir, *Bell System Tech. J.* **34**, 105 (1955).

[99f] S. V. Airapetyants and M. S. Bresler, *Fiz. Tverd. Tela* **1**, 152 (1959) [translation: *Soviet Phys. — Solid State* **1**, 134 (1959)].

[100] I. Isenberg, B. R. Russel, and R. F. Greene, *Rev. Sci. Instr.* **19**, 685 (1948).

[101] J. Volger, *Phys. Rev.* **79**, 1023 (1950).

[102] V. Frank, *Appl. Sci. Research* **B3**, 129 (1953).

[103] W. F. Flanagan, P. A. Flinn, and B. L. Averbach, *Rev. Sci. Instr.* **25**, 593 (1954); **26**, 233 (1955).

x-direction, with H reduced to approximately 20% at the contacted ends of the specimen, the V_m/V_H was found to be 0.74 for a specimen with $L/W = 1.5$, as compared to 0.85 when the magnetic field is uniform.

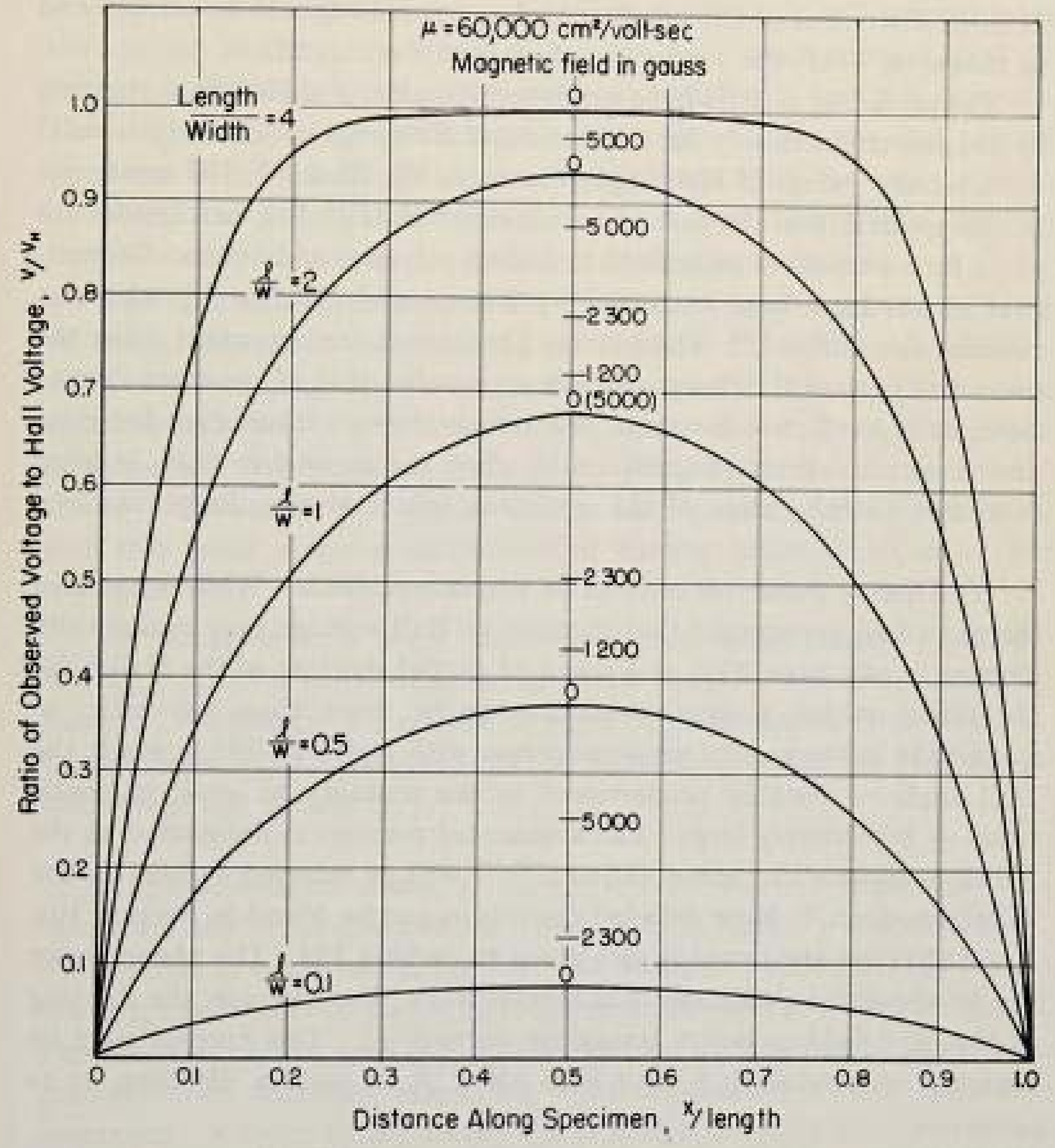

FIG. 4. Effect on Hall voltage, as a function of position along specimen, of transverse currents caused by electrode shorting at ends. Curves are drawn for the case of small Hall angles ($H \sim 0$) for various length-width ratios. At larger Hall angles (increased magnetic fields) the reduction in Hall voltage is less (larger V/V_H), as is evidenced by the points calculated for these cases when probes are located at $x/l = 0.5$.

Further information on the effects of inhomogeneous magnetic fields is available in the article by Koppe and Bryan.[104] These authors also discuss the effects of macroscopic inhomogeneities in the samples. The special case of porous media has been treated by Juretschke *et al.*[105] For a more detailed discussion of the effects of inhomogeneities, both in sample and in magnetic field, the reader is referred to Section 27.

The preceding articles have examined the effects of Hall field shorting by the end contacts only for cases of *small Hall angles*. An analysis valid for arbitrary values of Hall angle was made by Wick.[106] His treatment is also applicable to the case of area-contact Hall probes, and results are given for a number of geometries including polygons and circles. Geometrical effects have been examined by Barron and MacDonald, who also consider size effects.[106a] These latter phenomena are important when the mean free path of the charge carriers approaches that of specimen dimensions, as in very pure metals at low temperatures. Interesting behavior also results at strong magnetic field when the mean free path is large compared to the radius of the cyclotron orbits of the charge carriers.

(2) *Effect of transverse currents on magnetoresistance.* While we saw in the preceding paragraphs that changes in Hall voltage may realistically amount to say 20 or 30% as a result of partial shorting of the Hall field, the effect on the magnetoresistance can be more than an order of magnitude in the case of semiconductors with high mobilities, where the Hall angle — roughly proportional to the mobility at weak magnetic fields — is relatively large. The augmented resistance, designated as the "Hall resistance" by some authors,[106,107] will be referred to here as the "Corbino effect." More detailed discussion can be found in Section 10*b*. A quantitative treatment is also given in Section 12*d*. The phenomenon can be visualized qualitatively in several ways. For example, the shorting of the Hall field causes a transverse current J_y. This current may be regarded as producing a Hall field along J_x in such a direction as to

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[104] H. Koppe and J. M. Bryan, *Can. J. Phys.* **29**, 274 (1951).

[105] H. J. Juretschke, R. Landauer, and J. A. Swanson, *J. App. Phys.* **27**, 839 (1956); E. Goldin and H. J. Juretschke, *Trans. AIME* **212**, 357 (1958).

[106] R. F. Wick, *J. Appl. Phys.* **25**, 741 (1954).

[106a] T. H. K. Barron and D. K. C. MacDonald, *Physica* **24** (Kamerlingh Onnes Conference, Leiden), S102 (1958).

[107] J. R. Drabble and R. Wolfe, *J. Electronics and Control* **3**, 259 (1957).

decrease J_x — or increase E_x, depending whether the driving power is of constant voltage or of constant current — and therefore being reflected as an increase in resistance. On the other hand, the result may be regarded as a more general magnetoresistance, where the boundary condition $J_y = 0$ is relaxed and the general form of (9.5) is applied. It will be seen later that in the limiting case where E_y is zero, the "magnetoresistance" in first approximation increases proportional to H^2 and does not saturate. This behavior can be anticipated from the simple physical picture presented in Section 4*b*. Without a Hall field to cancel the Lorentz force, the charge carriers will be deflected at angles increasing with H, leading to a monotonic increase in resistance with magnetic field.

A mathematical treatment of the effect of shorting at the end contacts on magnetoresistance is presented by Drabble and Wolfe, and results are given for length/width ratios greater than two.[107] The results illustrate quite vividly the problem that arises when the Hall angle becomes appreciable. Experimental findings are given by Broom for GaAs and InSb.[108] Frederikse and Hosler discuss the "geometry effect" in InSb, and also point out the seriousness of having inhomogeneities in the material.[109]

The measurements show that when measuring magnetoresistance in a high-mobility semiconductor, all lateral contacts should be extremely minute. Welded platinum wires of 3-mil diameter appeared permissible. On the other hand, soldered probes of 0.25-mm diameter, or bridge-type arms were not satisfactory.

(3) *Nongeometrical causes of transverse currents.* An important consideration here is the effect of inhomogeneities in impurity concentrations. This matter is examined in considerable detail in Section 27. It is seen that large influences on the galvanomagnetic effects can result from spatial variations in carrier density. Either random fluctuations or monotonic gradients can seriously affect the field dependence of the magnetoresistance. A contributing factor which occurs for most nonuniformities is a variation in current distribution in the specimen with magnetic field. This rearrangement of the flow lines results from the boundary conditions

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[108] R. F. Broom, *Proc. Phys. Soc. (London)* **71**, 500 (1958).

[109] H. P. R. Frederikse and W. R. Hosler, *Phys. Rev.* **108**, 1136 (1957).

and the fact that **E** and **J** are, respectively, lamellar and solenoidal vector point functions.

In view of the complications that occur, it behooves the investigator to check carefully the homogeneity of the specimens by measuring potentials between various contacting points. Other techniques such as resistivity probing, thermoelectric probing[94] and photoconductive measurements with a traveling light spot are often employed. The latter method is often used to measure diffusion lengths of nonequilibrium charge carriers.[110-112] An unnatural variation of the photovoltage with position of the light spot could be indicative of nonuniformity in the specimens.[112a]

Analyses of *bulk* photovoltaic phenomena (changes in impurity concentration occur over distances large compared with nonequilibrium charge carrier diffusion lengths) and of *barrier-layer* photovoltaic phenomena (abrupt changes in impurity density) have been made by Tauc.[113] He also outlines a compensation method, due to Frank,[114] for measuring the bulk photovoltage, which provides a highly sensitive method for detection of inhomogeneities in a semiconductor. Additional investigations of bulk photoeffects in inhomogeneous semiconductors, including phenomena at an illuminated *p-n* junction, were done by Cox.[114a] Experimental data, taken on germanium filaments, are also supplied.

Thus far, we have dealt principally with transport by a single charge carrier, choosing to introduce multiband contributions later. At this point, however, it is important to consider the simultaneous presence of electrons and holes — as, for example, in intrinsic semiconductors. The boundary condition that J_y vanish does not imply that the electron and hole currents are individually zero, but rather that

$$j_y^e + j_y^h = 0. \tag{9.18}$$

In the magnetic field, both holes and electrons are directed to the same side of the specimen, producing concentration gradients and diffusion

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[110] F. S. Goucher, *Phys. Rev.* **81**, 475 (1951).

[111] L. B. Valdes, *Proc. I. R. E.* **40**, 1420 (1952).

[112] T. S. Moss, *Proc. Phys. Soc.* (*London*) **B66**, 993 (1953).

[112a] J. Arthur, W. Bardsley, A. Gibson, and C. Hogarth, *Proc. Phys. Soc.* (*London*) **B68**, 121 (1955).

[113] Jan Tauc, *Revs. Modern Phys.* **29**, 308 (1957).

[114] H. Frank, *Czechoslov. J. Phys.* **6**, 433 (1956).

[114a] C. D. Cox, *Can. J. Phys.* **88**, 1328 (1960).

currents. The generation and recombination of carriers in the bulk may often be neglected and one can consider hole-electron pairs to be continuously generated at one surface and continuously recombined at the other. A treatment of galvanomagnetic properties must therefore include a description of these diffusion processes. Additional parameters of interest are diffusion constants, sample width, lifetime of excess carriers in the bulk, and surface recombination velocities.

An investigation of these phenomena was begun by Welker[115] after he noticed a frequency effect in ac Hall measurements[116] on mixed (i.e., both electrons and holes are present) semiconductors. This behavior comes about as a result of the hole-electron accumulations no longer being able to follow the oscillations of the applied field.

When the lifetime of the excess carriers is very low, their concentrations are everywhere near equilibrium and therefore concentration gradients and diffusion currents are not significant. The effect is largest in the case of infinite lifetimes. This example has been considered by Fowler.[117]

Further analyses of the problem have been given by Landauer and Swanson[118] and by Banbury *et al.*[119] Explicit results are available for the magnitudes of the corrections to the Hall fields for most cases of interest. Another complication is the fact that the measured transverse potentials can be affected by the nature of the probes. In many cases rectifying barriers will exist at the contacts and floating potentials[120] will be measured along with the other contributions to the Hall voltage.

The prominence of the effects discussed above is decreased when the departures of the charge carrier concentrations from equilibrium are reduced, as can be accomplished by increasing surface recombination rates. Thus the surfaces of the specimens should be abraded rather than etched. Attempts should also be made to make ohmic contacts at the voltage and current probes. This usually requires a specific treatment

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[115] H. Welker, *Z. Naturforsch.* **6a**, 184 (1951).

[116] H. Welker, *L'Onde Elec.* **30**, 309 (1950).

[117] R. H. Fowler, "Statistical Mechanics," p. 428, Cambridge Univ. Press, Cambridge, 1936.

[118] R. Landauer and J. Swanson, *Phys. Rev.* **91**, 555 (1953).

[119] P. C. Banbury, H. K. Kenisch, and A. Many, *Proc. Phys. Soc.* (*London*) **A66**, 753 (1953).

[120] J. Bardeen, *Bell System Tech. J.* **29**, 469 (1950).

for each type semiconductor. Certain general procedures have been described by Waltz[121] and the characteristics of certain metal contacts on germanium have been given by Bocciarelli.[122] Thermocompression bonding techniques have been described by Anderson *et al.*[123] In many cases useful information is given in the sections on "Experimental Details" in articles dealing with measurements of electrical properties of given semiconductors.[124–126a]

The careful investigator will usually repeat the galvanomagnetic measurements for several different surface conditions, including the extremes of sand blasting and etching.[126] Checks between symmetrical pairs of contacts are also desirable. If all of these operations reveal negligible differences, then it is likely that the effects that have just been discussed are not significant.

Another important phenomenon which can usually be detected when measurements are taken under different surface conditions is that of surface conduction. The existence of such processes can lead to serious errors in the determination of galvanomagnetic coefficients.[94,127]

When measuring high-lifetime materials, it is desirable to take precautions to ensure that errors are not introduced through occurrence of minority carrier injection at the current contacts. If possible, all voltage probes should be located well over a diffusion length from these contacts.

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[121] M. C. Waltz, *Bell Lab. Record* **33**, 260 (1955).

[122] C. V. Bocciarelli, *Physica* **20**, 1020 (1954).

[123] O. Anderson, H. Christensen, and P. Andreatch, *J. Appl. Phys.* **28**, 923 (1957).

[124] For germanium, see for example: J. R. Haynes and W. Shockley, *Phys. Rev.* **81**, 835 (1951); F. J. Morin and J. P. Maita, *ibid.* **94**, 1525 (1954); W. C. Dunlap, Jr., *ibid.* **96**, 40, (1954); C. Herring, T. Geballe, and J. Kunzler, *ibid.* **111**, 36 (1958).

[125] For silicon, see, for example: F. J. Morin and J. P. Maita, *Phys. Rev.* **96**, 28 (1954); G. W. Ludwig and R. L. Watters, *ibid.* **101**, 1699 (1956); Donald Long, *ibid.* **107**, 672 (1957); A. Gorodetskii, V. Mel'nik, and I. Mel'nik, *Fiz. Tverd. Tela* **1**, 173 (1959) [translation: *Soviet Phys.-Solid State* **1**, 153 (1959)].

[126] E. H. Putley and W. H. Mitchell, *Proc. Phys. Soc.* (*London*) **72**, 193 (1958).

[126a] For certain III–V compounds, see, for example: (InP) W. Reynolds, M. Lilburne, and R. Dell, *Proc. Phys. Soc.* (*London*) **71**, 416 (1958); (GaAs) J. M. Whelan and G. H. Wheatley, *Phys. and Chem. Solids* **6**, 169 (1958); R. K. Willardson, *J. Appl. Phys.* **30**, 1158(1959); (AlSb) H.-J. Henkel, *Z. Metallk.* **50**, 51 (1959).

[127] See, for example, the results of Frederikse *et al.*: H. P. R. Frederikse, W. R. Hosler, and D. E. Roberts, *Phys. Rev.* **103**, 67 (1956).

Another precaution is to maintain low current densities at the contacts. As a matter of fact, the careful experimentalist will determine galvanomagnetic voltages as a function of specimen current, generally over ranges of 1 or 2 decades. If the voltages are not linear with the currents, then the reason must be ascertained. Noticeable heating of the specimen must of course be avoided. In addition, the electric field must not be so high that the specimen becomes non-ohmic. This happens when the charge carrier receives more power from the field than it can dissipate readily by the usual interaction with the lattice. Such a situation is referred to as the "hot-carrier case." Another non-ohmic phenomenon is charge carrier multiplication through avalanche or breakdown processes. These effects are discussed briefly in Section 32. In most semiconductors they are not encountered at electric fields low enough to preclude heating, except at temperatures of liquid helium.

Hunter *et al.*[127a] have discussed a scheme for determining charge-carrier lifetimes from the changes in Hall coefficient produced by deviations in carrier concentration from equilibrium. It was suggested that the method,may be of especial advantage in measuring very short lifetimes, i.e., those of the order of 0.1 μ sec.

One may expect that the influence of transverse diffusion currents will be especially great on the magnetoresistance effect. This premise has been verified by the experiments done on the "magnetic barrier layer" phenomena. In these experiments, dissymmetry is created by using different surface treatments on the two sides of the specimen normal to the Hall field.[128] The roughened surface (Fig. 5) has a high recombination velocity s, while the other surface is etched to produce low s. Thus, for a preferred direction of current, recombination is facilitated. With germanium, resistance ratios of the order of 10 were readily observed for magnetic fields of 10 kgauss.

The effects of nonequilibrium carrier concentrations on galvanomagnetic and thermomagnetic effects in semiconductors have been studied

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[127a] L. Hunter, E. Huibregtse, and R. Anderson, *Phys. Rev.* **91**, 1315 (1953).

[128] See E. Weisshaar and H. Welker, *Z. Naturforsch.* **8a**, 681 (1953); **9a**, 184 (1954); O. Madelung, *Naturwissenschaften* **42**, 406 (1955); O. Madelung, L. Tewordt, and H. Welker, *Z. Naturforsch.* **10a**, 476 (1955); E. Weisshaar, *Z. Naturforsch.* **10a**, 488 (1955); T. S. Moss *in* "Semiconductors and Phosphors" (M. Schön and H. Welker, eds.), p. 109. Interscience, New York, 1958.

quite generally by Pikus. Both the cases of weak magnetic fields[129] and strong magnetic fields[129a] are included. The expression for weak-field

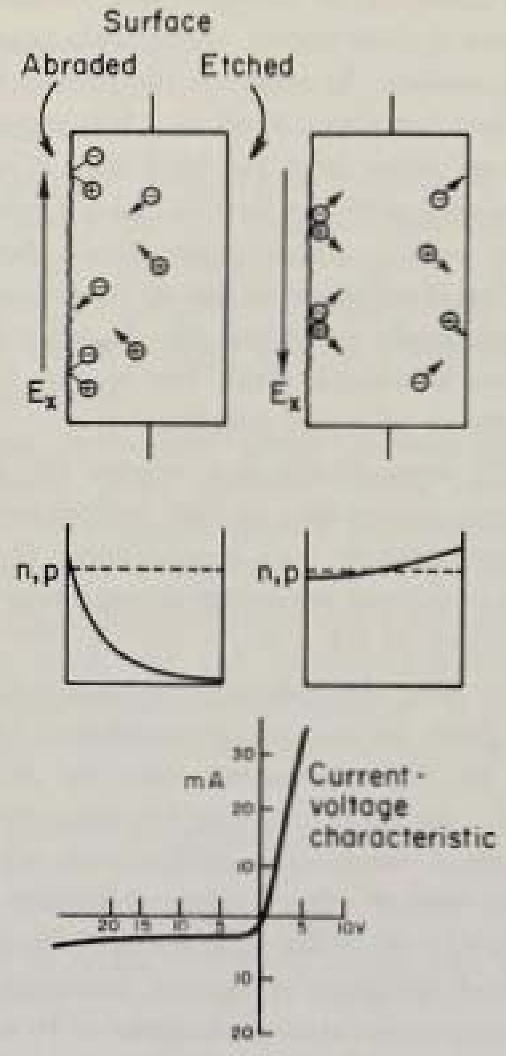

FIG. 5. Principle of the magnetic barrier layer. The abraded surface has a high recombination velocity, facilitating recombination of electrons and holes. Thus, for a current direction such that the charge carriers are deflected toward the abraded surface, the carrier concentration is impoverished and the current is reduced (after Madelung[128]).

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[129] G. E. Pikus, *J. Tech. Phys. U.S.S.R.* **26**, 22 (1956) [translation: *Soviet Phys.-Tech. Phys.* 1, 17 (1956)].

[129a] G. E. Pikus, *J. Tech. Phys. U.S.S.R.* **26**, 36 (1956) [translation: *Soviet Phys.-Tech. Phys.* 1, 32 (1956)].

magnetoresistance includes linear terms in H if the surface recombination effects are different on the two surfaces normal to the Hall field. If the width of the sample is small compared to a diffusion length, the linear term may become relatively large at weak magnetic fields.

The influence of nonequilibrium charge carrier distributions on thermoelectric and thermomagnetic effects is also considered by Pikus. The effects include electrical resistance in a temperature gradient, the Ettingshausen effect, and thermal conduction by electrons and by phonons. A discussion is also given of the measurement of thermoelectric power by point probes and how the results can depend on the radius of the tip of the probe due to a buildup of excess charge carriers at the point contact due to a large temperature gradient. This problem has been investigated both theoretically and experimentally by Tauc and Trousil.[129b] Additional discussion of nonisothermal effects is given in Section 29.

The large effects on magnetoresistance resulting from assymmetry in surface recombination rates in intrinsic semiconductors has been examined by several investigators[129c] as a means for measuring surface recombination velocities and bulk lifetimes. Under certain conditions, quantitative agreement with results of other measurement techniques was obtained.

Another method for obtaining magnetoresistance data under the conditions of zero Hall field is to measure a mixed semiconductor (i.e., both electrons and holes are present) at the Hall effect null. In many p-type materials it is possible to choose a temperature and a magnetic field H_1 such that in Eq. (9.5) the absolute values of

$$\overset{\text{electrons}}{\sigma_{xy}(H_1)} = \overset{\text{holes}}{\sigma_{xy}(H_1)}. \tag{9.18}$$

Under these conditions E_y vanishes, and if $\sigma_\infty^e = \sigma_\infty^h$, the magnetoresistance will fail to saturate. For the more general case where $\sigma_\infty^e \neq \sigma_\infty^h$, consult the equations given in Section 16.

[129b] J. Tauc and Z. Trousil, *Czechoslov. J. Phys.* **3**, 120 (1953). See also Section 7 of reference 113 and literature cited therein; J. Tauc, *Izvest. Akad. Nauk S.S.S.R.* **20**, 1479 (1956) [translated by Columbia Technical Translations: Ia. Tauts, *Bull. Acad. Sci. U.S.S.R.* (*Phys. Ser.*) **20**, No. 12b, 1357 (1956)].

[129c] See, for example: S. A. Poltinnikov and L. S. Stil'bans, *J. Tech. Phys. U.S.S.R.* **27**, 30 (1957) [translation: *Soviet Phys. — Tech. Phys.* **2**, 23 (1957)]. O. V. Sorokin, *J. Tech. Phys. U.S.S.R.* **27**, 2774 (1957) [translation: *Soviet Phys. — Tech. Phys.* **2**, 2572 (1957)]. A Rzhanov, I. Arkhipova, and V. Bidulia, *J. Tech. Phys. U.S.S.R.* **28**, 1051 (1958) [translation: *Soviet Phys. — Tech. Phys.* **3**, 978 (1958)].

(*4*) *Temperature gradients.* The interpretation of galvanomagnetic data in terms of fundamental parameters of the semiconductor is greatly simplified if isothermal conditions are assumed. However, this state of affairs will, for ordinary measurement techniques, exist precisely only for materials with very special characteristics — as can be seen from Eqs. (3.5). The other possibility of achieving isothermal environment is an experimental setup involving complicated equipment for adding or removing heat from various faces of the specimen so as to reduce the temperature gradients. For these reasons, many investigators have resorted to the use of ac measurements. These usually involve expensive instrumentation equipment. A promising alternative is the Dauphinee-Mooser scheme in which square wave currents of alternating polarity are produced by a special chopper.[130] This arrangement combines the desirable features of both the dc and the ac techniques.

In dc measurements, nonisothermal contributions can arise from thermoelectric and thermomagnetic effects. These are sometimes considered separately as follows:

(i) *Peltier effect:* This phenomen is associated with heat transport across a junction, and it therefore causes a temperature gradient to exist along the sample in which there is an electric current. The Peltier effect occurs because of a nonzero differential thermoelectric power between the sample and the metal contacts.

(ii) *Nernst effect* (transverse): A longitudinal temperature gradient or heat current gives rise to a transverse electric field in a crossed magnetic field. Designation of this phenomenon by *Nernst* effect is used by a number of authors,[3, 31, 46, 61, 131 – 134] although others prefer the term *Ettingshausen-Nernst.*[90, 135, 135a] The situation is complicated by the fact that a longitudinal temperature gradient due to a longitudinal current in a transverse magnetic field is also known as the Nernst effect.

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[130] T. M. Dauphinee and E. Mooser, *Rev. Sci. Instr.* **26**, 660 (1955).

[131] R. Fieschi, S. de Groot, and P. Mazur, *Physica* **20**, 259 (1954). These authors use the term *Ettingshausen-Nernst* to designate the thermoelectric power in a magnetic field.

[132] See p. 60 of the article by O. Madelung in "Handbuch der Physik" (S. Flügge, ed.), Vol. 20, p. 1. Springer, Berlin, 1957.

[133] See p. 230 of the article by Blatt.[44]

[134] P. J. Price, *Phys. Rev.* **102**, 1245 (1956).

[135] R. G. Chambers, *Proc. Phys. Soc.* (*London*) **A65**, 903 (1952).

[135a] Reference 9, p. 209.

(iii) *Righi-Leduc effect:* This designation refers to the transverse temperature gradient resulting from a longitudinal temperature gradient, or heat current, in a crossed magnetic field.

(iv) *Ettingshausen effect:* In this case, a longitudinal electric current produces a transverse temperature gradient in a crossed magnetic field.

Both (iii) and (iv) will produce spurious Hall voltages because of the differential thermoelectric power of the specimen against the Hall probe materials. This contribution could be minimized by making the Hall leads of the same material as the specimen; but such a procedure is hardly feasible in the case of semiconductors.

In general, measurements are taken for both directions of magnetic field and of sample current. In this way, some of the spurious effects can be reduced.[95] A point of caution should be injected, however, inasmuch as there is a tendency among some experimenters to assume that since the Nernst and the Righi-Leduc phenomena depend on temperature gradients, they will remain unchanged with reversal of the electric current. This is the state of affairs if the source of the thermal gradients is, for example, nonelectrical, or results from nonuniform joule heating either in the specimen or at the end contacts. Any contribution from the *Peltier* effect, or related irreversible phenomena, will, on the other hand, change sign upon reversal of the electric current.

Other safeguards involve the use of isothermal baths for the specimen and of observing whether there are discernible time lags in the approach to the steady state voltage as the current is reversed. These would be evidence of a reversal in thermal gradients. The magnetic field is usually reversed by switching the polarity of the dc into the electromagnet. It has been found, however, that unless demagnetization procedures were used, differences in $|\mathbf{H}|$ of about 2% existed after such a reversal in a 12-inch magnet.[136] Hence for precise measurements, the magnitude of $\mathbf{H}$ is continuously monitored by a nuclear magnetic resonance gaussmeter. Upon reversal of the magnet current, an adjustment is made to maintain $|\mathbf{H}|$ unchanged.

An important question is when are differences between isothermal and adiabatic galvanomagnetic coefficients significant? This matter is discussed in Section 29, where it is seen that an important consideration

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[136] R. T. Bate, Unpublished findings at the Battelle Memorial Institute.

is the "thermoelectric figure of merit" Z^*, defined in terms of thermoelectric power, temperature, thermal conductivity, and electrical resistivity as follows:

$$Z^* \equiv \alpha^2 T/\kappa\rho. \tag{9.19}$$

This point was recognized by Chambers[135] who showed that expressions for adiabatic effects simplified considerably when the square of the thermoelectric power was much smaller than the Lorenz number.

In view of (9.19) it appears likely that in materials with the diamond lattice or zincblende structure, where the lattice contribution to κ is relatively large, the corrections to the isothermal equations for specimens measured under adiabatic conditions may be relatively small, at least in the weak-magnetic-field region. This was found to be the case for the Hall coefficient according to calculations by Stil'bans[137] and by Johnson and Shipley.[138] In the latter article, the results for materials such as silicon and germanium showed differences well under 1% for a wide variation of temperatures and carrier concentrations.

The problem is much more complicated when the weak magnetic field approximation is not applicable, for then expansions utilizing powers up to H^2 are no longer adequate. The transport integrals must be evaluated exactly. In addition, various product terms occur which cannot be neglected. For this reason, certain investigators have carried out experiments on high-mobility semiconductors under the conditions that E_y and $\partial T/\partial y$ vanish, as is achieved with the Corbino disk, rather than for conditions such that J_y and q_y vanish.[139]

10. Other "Hall" and Magnetoresistance Phenomena

a. Various Hall Phenomena

(*1*) *Quadratic "Hall" effect.* Discussions of quadratic contributions to the Hall effect can be found in early articles by Kohler[140] and by

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[137] L. S. Stil'bans, *J. Tech. Phys. USSR* **22**, 77 (1952).

[138] V. A. Johnson and F. M. Shipley, *Phys. Rev.* **90**, 523 (1953).

[139] A. C. Beer, J. A. Armstrong, and I. N. Greenberg, *Phys. Rev.* **107**, 1506 (1957)

[140] M. Kohler, *Ann. Physik* **20**, 891 (1934).

Shoenberg.[141] Some of the equations need to be modified as a result of the Kohler-Onsager relations — which at that time had not been established. However, Shoenberg does present equations for a quadratic "Hall" effect which results from a component of **E** being perpendicular to **J** in a crossed magnetic field, because of anisotropy of the medium. Such a situation can occur, for example, if the galvanomagnetic tensor component ρ^0_{3211} does not vanish. This is true in a crystal such as bismuth [see Eq. (8.36)], but is not the case for the higher symmetry of the cubic system [see Eq. (8.12b)] when the coordinate axes are along the axes of cubic symmetry.

(2) *Less conventional "Hall" phenomena.* In the preceding paragraph, we abandoned the requirement that the Hall field be an odd function of H,[71] maintaining only the condition that $\mathbf{E}_{\text{Hall}}$, $\mathbf{J}$, and $\mathbf{H}$ be mutually orthogonal. We saw that in anisotropic media, terms involving even powers of H could contribute to E_{Hall} via the galvanomagnetic tensor ρ^0_{ijkk}. If one goes a step further, and requires only that $\mathbf{E}_{\text{Hall}}$ be normal to $\mathbf{J}$, then quadratic Hall terms can occur even in *isotropic* media. This general definition is adopted by Kao and Katz.[67c] If $\mathbf{E}_{\text{meas}}$ is normal to $\mathbf{J}$, they call the dependence $E_{\text{meas}}(\mathbf{J}, \mathbf{H})$ a Hall effect; if $\mathbf{E}_{\text{meas}}$ is parallel to $\mathbf{J}$, then $E_{\text{meas}}(\mathbf{J}, \mathbf{H})$ is called the magnetoresistance. This extension in definition is illustrated by the "planar Hall effect," which was investigated by Goldberg and Davis.[142] It has also been referred to as the "pseudo-Hall-effect."[143]

Consider an isotropic solid, or a cubic crystal with axes along the coordinate axes, with **J** along the x_1-direction and **H** in the $x_1\,x_2$-plane at an angle φ with **J** (Fig. 6).

It is apparent that

$$E_2 = (\rho^0_{2112} + \rho^0_{2121}) J H_1 H_2 = 2\rho^0_{2121}\, J H^2 \sin\varphi \cos\varphi, \tag{10.1}$$

$$E_2 = \rho^0_{2121}\, J H^2 \sin 2\varphi. \tag{10.2}$$

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[141] D. Shoenberg, *Proc. Cambridge Phil. Soc.* **31**, 271 (1935).

[142] C. Goldberg and R. E. Davis, *Phys. Rev.* **94**, 1121 (1954).

[143] K. M. Koch, *Z. Naturforsch.* **10a**, 496 (1955).

The ρ^0_{2121} can be expressed in terms of the constants in (8.30) by use of relations (8.31b) to give

$$E_2 = (c\rho_0/2)JH^2 \sin 2\varphi. \tag{10.3}$$

This is the expression given by Goldberg and Davis. They also show another orientation of the cubic system which can be chosen so as to yield the constant d, in the form of the coefficient $(c + d)\rho_0/2$. Since d, which vanishes in an isotropic crystal, is small in most cubic systems, it is not very accurately determined by means of the above coefficient. To obtain improved accuracy, the authors suggest a procedure which involves determination of the coefficient of the quadratic term in the Hall coefficient discussed in Subsection (1) above for a special orientation of crystallographic and coordinate axes.

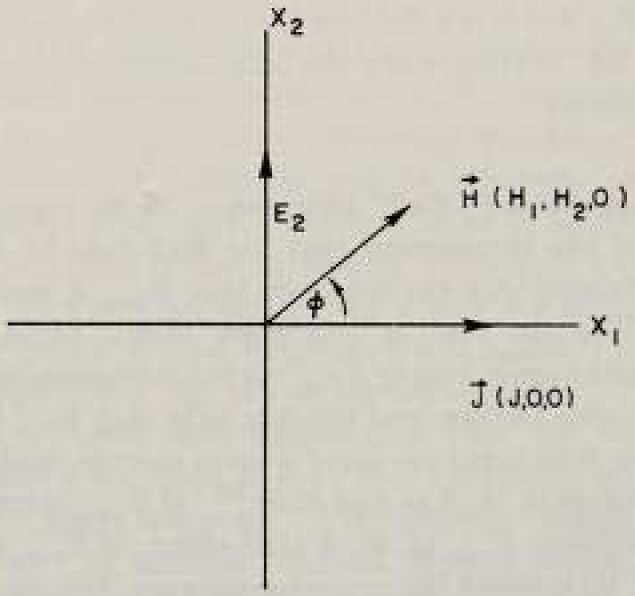

FIG. 6. Illustration of planar Hall effect.

It is important to note that the simple form of Eqs. (10.1) – (10.3) applies only if there is sufficient symmetry — as is true, for example, in the cubic system and the trigonal system which were considered previously — so that coefficients such as ρ^0_{21}, ρ^0_{211}, ρ^0_{2111}, ρ^0_{2122}, vanish.

(3) *Concerning the definitions of the Hall effect.* It is apparent that various definitions of the Hall effect found in the literature can lead to

different results, depending upon the contributions entering through the tensors ρ^0_{iklm}. When the results of the measurements are interpreted in terms of fundamental characteristics of the semiconductors, the fields resulting from the different resistivity tensor components will need to be separated. The experimenter must therefore isolate the effects which are even in H from those which are odd. In view of this, there would appear to be advantages in defining the Hall effect as the antisymmetrical part of the resistivity tensor $\rho_{ik}(\mathbf{H})$, as is done by Jan.[144,144a] As a result of the Kohler-Onsager relations, the antisymmetrical part is that part which changes its sign upon reversal of **H**. The Hall effect thus defined is represented by the Hall vector $\mathbf{R}(\mathbf{H})$ in Eq. (8.21). It is approximated to first order in H by the ρ^0_{ikl} tensors.

The *magnetoresistance* effects are then defined as those contributions arising from the *symmetric* part of $\rho_{ik}(\mathbf{H})$. It thus follows that magnetoresistance is an even function of **H** — as is assumed in Section 11*b*.

Regardless of the definition adopted for the Hall effect, it is necessary for the person analyzing the data to recognize what basic effects are included in each measurement. In isotropic media where $\mathbf{E}_{\text{meas}}$, **J**, and **H** are mutually orthogonal, there is no ambiguity. This, as we have seen, is not true when anisotropy exists.

In view of the problems discussed above, there is a growing tendency to avoid use of the terms Hall effect and magnetoresistance when dealing with weak-magnetic-field measurements in anisotropic crystals. Instead, the basic galvanomagnetic coefficients are determined. In those cases where stronger magnetic fields are used, however, expansions of the components of $\boldsymbol{\rho}(\mathbf{H})$ in power series of H are not possible (see Section 6*b*). Then the galvanomagnetic coefficients as defined in Eq. (8.20) do not exist. [For additional discussion of Hall phenomena, see Section 11*b* (2).]

b. Corbino Magnetoresistance, Magnetoconductivity

We have seen that the ordinary magnetoresistance is measured under the boundary condition that the transverse current be zero [Eq. (9.10)]. It is possible, however, to arrange a shorting of the Hall voltage, so that

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[144] See reference 1, p. 13.

[144a] Such a point of view was adopted by Logan and Marcus. See J. K. Logan and J. A. Marcus, *Phys. Rev.* **88**, 1234 (1952).

the transverse electric field is zero. Then, for isotropic media and isothermal conditions, the following form of Eq. (9.5) applies:

$$J_x = \sigma_{xx}(H)E_x \qquad (10.4)$$

$$H \equiv H_z, \quad E_y = 0.$$

$$J_y = -\sigma_{xy}(H)E_x \qquad (10.5)$$

It is seen from the first equation that a measurement of the ratio of J_x and E_x yields directly the conductivity tensor component $\sigma_{xx}(H)$, since the nondiagonal component $\sigma_{xy}(H)$ enters only via the shorting current J_y.

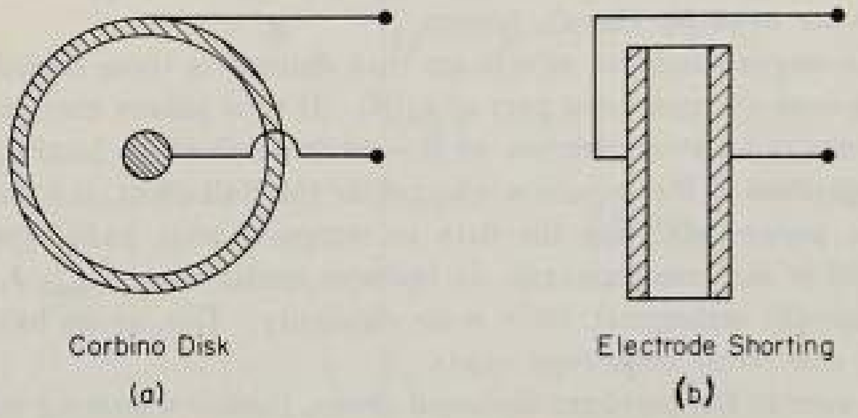

Fig. 7. Mechanisms to produce shorting of the Hall voltage.

In effect, then, we are determining directly the *magnetoconductivity* of the specimen when we measure the ratio of currents appearing in the equation below:

$$\frac{\sigma(H)-\sigma(0)}{\sigma(0)} = \frac{J_x}{(J_x)_{H=0}} - 1 = \frac{\sigma_{xx}(H)}{\sigma_{xx}(0)} - 1, \qquad \begin{matrix} E_y = 0 \\ E_x = \text{const.} \end{matrix} \qquad (10.6)$$

The above relation follows at once from (9.8) and (10.4).

The most effective arrangement for shorting the Hall field, so that (10.4) and (10.5) apply, is by use of the Corbino disk geometry,[145] having highly conducting metal contacts at the center and along the periphery (Fig. 7a).

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[145] After O. M. Corbino who investigated the circulating secondary currents in a bismuth disk carrying a primary radial electric current in a magnetic field.[146]

Originally, much interest centered around the circulating current, called the Corbino effect. It has been determined by measuring the current induced in a coil of wire placed parallel to the disk when the radial current was reversed about 20 cps.[147] Further determinations of the Corbino effect have been done by measuring the torque exerted on the current carrying disk by the magnetic field. Adams[146, 147] established the relationship between the Corbino effect and the Hall effect.

With the use of Eq. (9.12) for weak magnetic fields, (10.5) may be put in the form

$$J_y = -R_0 \sigma_0 H J_x, \qquad \omega\tau \ll 1 \tag{10.7}$$

where R_0 and σ_0 are the zero magnetic field Hall coefficient and conductivity, respectively. The total currents per unit thickness, circulating and radial, are given in terms of the current densities as follows:

$$I_c = \int_{r_1}^{r_2} J_y(r)\,dr, \qquad I = \int_0^{2\pi} J_x(r)\,r\,d\theta \tag{10.8}$$

where r_1 and r_2 are the inner and outer radii of the disk, and where we have associated subscripts x and y with the radial and angular components, respectively, of **J**.

Expression (10.7) thus becomes

$$I_c = -\frac{R_0 \sigma_0}{2\pi} HI \log \frac{r_2}{r_1}, \tag{10.9}$$

an expression identical with that of Adams. The results given by Adams for the Corbino magnetoresistance are

$$(\Delta\rho/\rho_0)_{E_y=0} = (R_0 \sigma_0 H)^2. \tag{10.10}$$

We shall see in Section 12 [Eq. (12.36)] that this result — derived before 1915 — applies precisely for the case of constant relaxation time, under

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[146] L. L. Campbell, "Galvanomagnetic and Thermomagnetic Effects: The Hall and Allied Phenomena," p. 125. Longmans, Green, New York and London, 1923.

[147] E. P. Adams, *Proc. Am. Phil. Soc.* **54**, 47 (1915).

which condition the Hall coefficient and resistivity are independent of H.

Because of the fact that Corbino disk measurements yield directly the value of $\sigma_{xx}(H)$ and that the Corbino magnetoresistance does not saturate at higher fields, it is a useful technique for studying the high-mobility semiconducting materials such as those of the III–V compounds.[130, 148]

Another means for shorting the Hall field is the use of area end contacts with small length-width ratios of the specimen (Fig. 7b). It was found by Welker that the Corbino magnetoresistance in n-type InSb at room temperature could be approximated very closely up to 3500 gauss by measurements on a specimen of such a rectangular geometry.[148] Experimental data on a number of length-width ratios for high purity InSb are also presented by Beer.[149] Shorting effects can also be produced in a long specimen by depositing thin metallic strips, or "shorting bars," across the surface.[150] This avoids the very low resistance which results from small length-width ratios. The width-length ratio necessary to approach the limiting case of the disk within a given percentage can of course be calculated by the theory of Wick[106] for an arbitrary Hall angle (magnetic field intensity).

We have seen that with $E_y = 0$, a "magnetoresistance" measurement yields the magnetoconductivity of the specimen. The distinction is that in magnetoresistance measurements the direction of the current **J** is fixed, independent of **H**, while the direction of **E**(H) is given by

$$E_y/E_x = \sigma_{xy}(H)/\sigma_{xx}(H) = \tan\theta$$

in the two-dimensional case. The quantity θ is the Hall angle. In magnetoconductance measurements, the direction of the electric field **E** is fixed, independent of **H**, and the direction of **J**(H) is given by

$$J_y/J_x = -\sigma_{xy}(H)/\sigma_{xx}(H) = -\tan\theta \qquad (10.12)$$

in the two-dimensional case. Thus, in the Corbino disk, where the equipotential curves are circles, the current lines spiral at the angles $-\theta$ with the normals to the equipotentials.[150]

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[148] H. Weiss and H. Welker, *Z. Physik* **138**, 322 (1954).
O. Madelung, *Naturwissenschaften* **42**, 406 (1955).

[149] A. C. Beer, *J. Electrochem. Soc.* **105**, 743 (1958).

[150] H. Welker and H. Weiss, *Solid State Phys.* **3**, 1 (1956); see especially pp. 38–39.

Use of the Corbino disk for magnetoconductivity determinations is feasible only for materials isotropic in the plane of the disk. Somewhat greater flexibility is possible with the large-area-contact specimens. Such a method has been used by Goldberg in studies on *n*-type germanium.[151,152] A serious experimental difficulty is the necessity for effective low-resistance contacts, and the importance of ascertaining that such conditions have been realized. The problem of contact resistance can be alleviated, however, by using an array of four closely spaced contacts at the center of an "infinite" sheet. Two of the contacts are for current connections and two are potential probes, permitting a measurement of potential difference with no voltage drop due to contact resistance. In addition, there are no shorting electrodes which might introduce other resistive losses. As long as the distance between the contacts is small compared to the distances to the boundaries, a measurement of the ratio $V(0)/V(H)$ at constant current, where V is the voltage difference between the potential probes, yields $\sigma_{xx}(H)/\sigma_{xx}(0)$. For purposes of subsequent identification we shall call this arrangement a "Corbino sheet."[152a] A conductivity measurement yields directly the magnetoconductivity inasmuch as the equipotential curves are unaffected — and therefore so is the direction of **E** — by the magnetic field. This is a result of the fact that the boundary condition which involves the Hall angle, and therefore the magnetic field, has been removed to infinity. The potential distribution is established simply by the solution of Laplace's equation subject to the boundary condition specifying fixed potentials at the two current contacts. The solution for the potential does not therefore involve the magnetic field, and the equipotential curves are unchanged as the field varies. A similar situation was encountered by Baker and Martyn in studies of electric currents in the ionosphere when circumstances were such that the Hall current built up no polarization.[152b] It has been pointed out by Barron and MacDonald that Corbino effects can occur when measuring very low resistivity metals (for example, very pure sodium at

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[151] C. Goldberg, *Bull. Am. Phys. Soc.* **2**, 65 (1957).

[152] C. Goldberg, *Phys. Rev.* **109**, 331 (1958).

[152a] The similarity between the Corbino sheet and the Corbino disk was first pointed out to the author by Dr. S. W. Kurnick ca. 1956. Salient features in the argument that $\sigma = \sigma_{xx}$ are due to Dr. F. J. Milford.

[152b] W. G. Baker and D. F. Martyn, *Phil. Trans. Roy. Soc. London* **A246**, 281 (1953); W. G. Baker, *ibid.* **A246**, 295 (1953).

liquid helium temperatures) where the contacting electrodes have higher resistivities than do the samples.[106a]

In spite of the experimental problems, Corbino measurements possess a number of attractive features as tools for studying high-mobility semiconductors — either through measurements of galvanomagnetic or thermomagnetic properties.[139]

11. Measurements on Anisotropic Materials

In anisotropic solids the transport properties are, in general, directional. Therefore it is necessary to specify the crystallographic directions of the currents and the applied magnetic fields. For example, magnetoresistance may be designated by $M^{[\mathbf{H}]}_{[\mathbf{J}]}$, which is defined phenomenologically as follows:

$$M^{[\mathbf{H}]}_{[\mathbf{J}]} = \left(\frac{\Delta\rho}{\rho_0}\right)^{[\mathbf{H}]}_{[\mathbf{J}]} = \frac{[\mathbf{E}(\mathbf{H}) - \mathbf{E}(0)] \cdot \mathbf{J}}{\mathbf{E}(0) \cdot \mathbf{J}}. \tag{11.1}$$

The subscript gives the direction of the current and the superscript that of the magnetic field. In general, measurements are taken for both directions of magnetic field so that $\mathbf{E}(\mathbf{H})$ is even in H. This point is discussed in more detail in Section 11*b*, to follow. It is convenient to define directional magnetoresistance *coefficients* for the crystallographic directions of the current and the magnetic field as follows:

$$\Upsilon^{mnp}_{hkl}(H) \equiv \frac{M^{mnp}_{hkl}(H)}{H^2} = \left(\frac{\rho(H) - \rho(0)}{\rho(0)H^2}\right)^{\mathbf{H}[mnp]}_{\mathbf{J}[hkl]} \tag{11.2}$$

For longitudinal magnetoresistance, where both sets of indices are identical, it is customary to write only the lower set. In the transverse magnetoresistance, $\mathbf{H}$ is of course normal to $\mathbf{J}$. In both cases

$$\mathbf{E}_{\text{meas}} \parallel \mathbf{J}. \tag{11.2a}$$

In a following section (Subsection *b*) we shall discuss other galvanomagnetic effects which are even in H, but where $\mathbf{E}_{\text{meas}}$ is normal to $\mathbf{J}$. These phenomena lead to quadratic "Hall" voltages.

a. Weak-Magnetic-Field Region

When $\omega\tau$ — or $\mu H/c$, where μ is the mobility of the charge carrier in the semiconductor — is less than unity, the transport coefficients can be expanded in a power series in H, as was seen in Section 8. Thus the electrical transport properties can be specified by the zero-magnetic-field resistivity and by the galvanomagnetic coefficients, as is seen in Eq. (8.20). This approach is preferable to that of classifying the experimental data in the categories of Hall coefficients and magnetoresistance, inasmuch as it avoids problems discussed in Section 10, which can arise as a result of joint contributions to transverse electric fields from Hall and magnetoresistance effects. The number of independent measurements necessary to determine the galvanomagnetic characteristics is dependent on the crystal symmetry, as was illustrated in Section 8.

In the weak-field region, the magnetoresistance varies as H^2. Using the notation of Eq. (11.2), we may therefore define weak-magnetic-field directional magnetoresistance coefficients as follows:

$$\Upsilon_{hkl}^{mnp} \equiv \frac{M_{hkl}^{mnp}}{H^2} = \lim_{H\to 0}\left[\frac{M_{hkl}^{mnp}(H)}{H^2}\right] = \lim_{H\to 0}\,[\Upsilon_{hkl}^{mnp}(H)]. \qquad (11.3)$$

These quantities appear frequently in the literature dealing with measurements on cubic systems.

(*1*) *Diamond cubic symmetry.* In cubic systems the zero-magnetic-field resistivity and the weak-field Hall coefficient are isotropic. The customary procedure is to determine the phenomenological weak-field magnetoresistance coefficients of Eq. (8.30). From this relation and (11.1), together with the requirement that $\mathbf{E}(\mathbf{H})$ be even in $\mathbf{H}$, one obtains

$$\frac{\Delta\rho}{\rho_0} \simeq bH^2 + c\,\frac{(\mathbf{J}\cdot\mathbf{H})^2}{J^2} + d\,\frac{J_1{}^2H_1{}^2 + J_2{}^2H_2{}^2 + J_3{}^2H_3{}^2}{J^2}, \qquad \omega\tau \ll 1. \qquad (11.3a)$$

Let us designate the direction cosines of $\mathbf{J}$ and $\mathbf{H}$ by ι_j and η_j ($j = 1, 2, 3$), respectively. Then $\iota_1\ \iota_2\ \iota_3$ are related to hkl by an ordinary normalization factor, as also are $\eta_1\,\eta_2\,\eta_3$ and mnp. The weak-field directional magnetoresistance coefficient can then be written

$$\Upsilon^{mnp}_{hkl} = b + c(\iota_j\,\eta_j)^2 + d(\iota_j^2\,\eta_j^2) \tag{11.4}$$

where the summation is from $j = 1$ to $j = 3$.

The above form was given by Pearson and Suhl,[79] who present data on n- and p-type germanium for the following directional coefficients:

$$\begin{aligned} &\Upsilon_{100} = b + c + d && \Upsilon^{010}_{100} = b \\ &\Upsilon_{110} = b + c + \tfrac{1}{2}d && \\ & && \Upsilon^{001}_{110} = b. \\ &\Upsilon^{\bar{1}10}_{110} = b + \tfrac{1}{2}d && \end{aligned} \tag{11.5}$$

The absence of the upper indices indicates that **H** has the same direction as **J**.

In studies of materials with the many-valley type band structure, it is of interest to know the ratio c/b reasonably accurately. For the model usually used for n-type germanium, where the energy surfaces are approximated by ellipsoids of revolution oriented along the [111] directions in k-space, the ratio should be -1. In order to determine this quantity from (11.5), three coefficients must be measured. The values of the weak-field plateaus of $\Delta\rho/\rho_0 H^2$ are required, necessitating measurements at small $\Delta\rho$ and at small values of magnetic field, which enters as the square. Usually extrapolations to $H = 0$ can amount to uncertainties of the order of several per cent. Thus substantial experimental problems are involved in obtaining c/b with good accuracy. Several useful techniques are described by Goldberg and Howard,[153] which have enabled them to secure improved results. Their findings give values of c/b within 3% of the theoretical figure for n-type germanium of 6×10^{15} carriers cm^{-3} and lower. Other measurements which have been used to provide information on magnetoresistance or magnetoconductance coefficients are the planar "Hall" effect (Section 10*a*) and magnetoconductivity (Section 10*b*). In the case of n-germanium, the planar "Hall" effect does not appear to yield the accuracy of the more direct methods.[153] The magnetoconductivity

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[153] C. Goldberg and W. E. Howard, *Phys. Rev.* **110**, 1035 (1958).

technique is not capable of providing results on the off-diagonal components of the conductivity tensor $\boldsymbol{\sigma}(\mathbf{H})$.

Higher order galvanomagnetic coefficients for n-type germanium have been evaluated by Mason *et al.*[64] Using Pearson and Suhl's data, they have determined four of the six coefficients of the H^4 terms. To determine the higher order terms in the Hall effect, they did measurements on two cylindrical specimens, one with its axis along the [100] direction and the other with the axis along a [110] direction. Hall coefficients were determined as a function of magnetic field up to 22 kgauss, using the following equation derived for the case of cylindrical geometry of diameter d:

$$R_H = \frac{V_H/d}{JH} = \frac{\pi}{4}\frac{V_H d}{IH} \tag{11.6}$$

where V_H is the measured Hall voltage, J the current density, and I the total current. The authors illustrate the orientations necessary to ensure a zero cross magnetoresistance effect on the Hall voltage for the cubic crystal. Using data obtained for such orientations, they determine the two coefficients for the first order correction to R_H [i.e., the coefficients of H^3 terms in the expression for $\rho(\mathbf{H})$] and three coefficients[153a] for the second-order correction (i.e., coefficients of the H^5 terms). From the results, it is seen that the correction terms on R_H are least if $\mathbf{H}$ is inclined only slightly from the appropriate cubic axis. Since several errors exist in the indices printed on Figs. 9 and 10 in the article, the text should be consulted.

(2) *Other crystal systems.* In the crystal systems of lower symmetry, many investigators prefer to carry out straightforward determinations of the galvanomagnetic coefficients rather than to use the conventional classifications, which can be ambiguous due to intermixing of voltages from Hall effect and magnetoresistance. Experimental arrangements which enable one to obtain the two resistivities and the 16 galvanomagnetic coefficients (through H^3 terms) in a crystal such as antimony have been outlined by Juretschke[80,82] and by Okada.[84] There are three sets of such measurements, defined by the orientations of $\mathbf{J}$ and $\mathbf{H}$ with

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[153a] According to the results obtained by Kao and Katz,[67c] four independent constants are necessary to specify these terms in the general case.

respect to the principal crystallographic axis **c**, usually directed along the z-coordinate axis. In each case the orientation of **J** with respect to the crystallographic axes is fixed by the cut of the specimen.

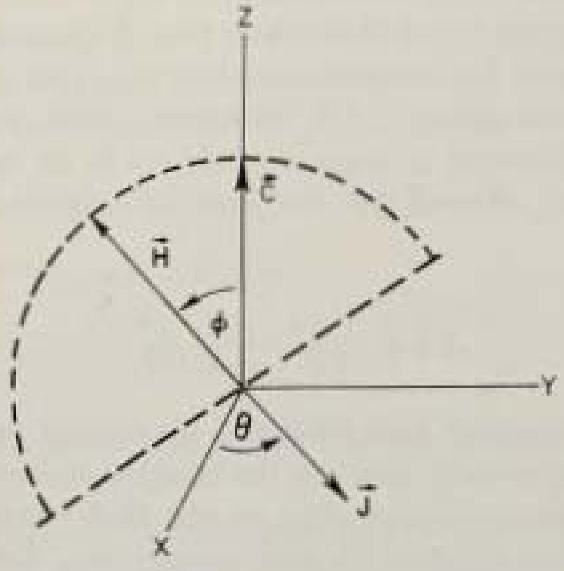

FIG. 8. Illustration of arrangement where **J** ⊥ **c** and **H** ⊥ **J**.

As an illustration, consider the schemes discussed by Juretschke[80] and outlined below.

Arrangement I. — **J** ⊥ **c**, **H** ⊥ **J**: If we designate by φ the angle between **H** and **c**, and by θ the angle between **J** and Ox — usually Ox or Oy is chosen coincident with one of the binary axes of the crystal — the relationship of the vectors is shown in Fig. 8. We may phrase the preceding clause more generally to include systems such as $3m$, which do not possess binary axes, by stating that the (y, z) or the (x, z) coordinate plane is made coincident with one of the three equivalent planes of reflection of the crystal. The angle θ, although arbitrary, is fixed by the cut of the specimen, and the angle φ is varied during measurements. The fields, which are measured in the directions indicated by the subscripts, are E_J, E_c, and $E_{c \times J}$. By a choice of θ appropriate for the particular coefficients desired, 12 of the 16 galvanomagnetic coefficients (including through the third order in the magnetic-field index) can be determined from this arrangement. Separation is accomplished by means of the φ-dependence. This takes account of the parity in H if φ is varied between zero and 2π.

Arrangement II. — $\mathbf{J} \parallel \mathbf{c}$, $\mathbf{H} \perp \mathbf{J}$: A measurement of $E_{\mathbf{J}}$ yields a value of $\rho^0_{3,1}$, in the notation of Eq. (8.36). Although measurements of $E_{\mathbf{H}}$ and $E_{\mathbf{J}\times\mathbf{H}}$ yield nothing which was not obtained in I, they are of interest as checks. The variable angle is that between $\mathbf{H}$ and Ox.

Arrangement III. — $\mathbf{J} \parallel \mathbf{H}$: An arbitrary angle with respect to $\mathbf{c}$ is necessary to include the determination of $\rho^0_{4,4}$. Both θ and φ are fixed, and the only degree of freedom is the reversal of $\mathbf{H}$. Relations connecting the unknown coefficients $\rho^0_{1,1}$, $\rho^0_{3,3}$, and $\rho^0_{4,4}$ through the angle φ with other coefficients are obtained for $E_{\mathbf{J}}$ and $E_{\mathbf{J}\times(\mathbf{c}\times\mathbf{J})}$. A measurement of $E_{\mathbf{c}\times\mathbf{J}}$ confirms preceding results. To separate the $\rho^0_{1,1}$, $\rho^0_{3,3}$, and $\rho^0_{4,4}$ it is desirable to have data for $\varphi = \pi/2$, $\varphi = 0$, and φ arbitrary (approximately $\pi/4$). Only one specimen needs to be fabricated for this arrangement, however, since the first two values of φ are obtained from the specimens used in arrangements I and II.

The specimen geometries are usually long thin cylindrical rods, or parallelepipeds of square cross section. Since the current direction must be *along* the length of the specimen, so as to avoid shorting effects discussed in Section 9*b*, each orientation of $\mathbf{J}$ with respect to the crystallographic directions requires a uniquely cut specimen. Thus three different specimens are necessary to determine the galvanomagnetic coefficients with the scheme outlined above. There is redundancy in the set of measurements, however, so that a number of cross checks between the different specimens are obtained.

A variation of the above scheme was used by Drabble[87a] in determining the 10 galvanomagnetic coefficients through H^2 terms. This procedure, in which the direction of $\mathbf{H}$ with respect to $\mathbf{J}$ was allowed to vary, has the advantage that only two specimens are required in the three arrangements. The system, again with the principal axis $\mathbf{c}$ in the coordinate direction Oz, is as follows:

Arrangement I ("$\perp$ specimen") — $\mathbf{J} \perp \mathbf{c}$, $\mathbf{H} \perp \mathbf{c}$: The current $\mathbf{J}$ is fixed at an arbitrary angle θ with respect to the x-axis, where the orientation is such that the (x, z) plane is coincident with one of the three equivalent planes of reflection of the crystal. The specimen is rotated about the **e** axis, which is equivalent to varying $0 \leqslant \varphi \leqslant 2\pi$, where φ is the angle between **H** and **J**. Measurements of E_J, E_e allow determination of ρ^0_{11} and four galvanomagnetic coefficients; and $E_{e\times J}$ gives a connecting relation.

Arrangement II ("⊥ *specimen*") – $\mathbf{J} \perp \mathbf{e}$, $\mathbf{H} \perp (\mathbf{e} \times \mathbf{J})$: Here the crystal is rotated about the $\mathbf{e} \times \mathbf{J}$ direction and therefore **H** is rotated throughout the **eJ**-plane. Measurements of E_J, E_e, and $E_{e\times J}$ provide four additional galvanomagnetic coefficients.

Arrangement III ("‖ *specimen*") – $\mathbf{J} \parallel \mathbf{e}$: The crystal is rotated about an axis normal to **e**, and **H** is of course perpendicular to this rotation axis. Measurement of E_J in this experiment allows determination of ρ^0_{33}, ρ^0_{3333}, and ρ^0_{3311}. Connecting relations are obtained if E is measured parallel to the rotation axis. The reader will note that the term *rotation axis* is used here to denote the axis about which the specimen is rotated during measurements in the magnetic field. This is in contrast to the specification of Drabble,[87a] who uses the term to denote the trigonal axis of the crystal.

In the case of *n*-type bismuth telluride, measurements with **J** along **e** were considered unreliable by Drabble *et al.*[87] since their specimens were highly subject to cracks in directions parallel to the cleavage planes. Therefore arrangement III was not used. This allowed determination of nine of the 12 coefficients, and only one specimen was required.

b. Magnetic Fields of Arbitrary Intensity

(1) General discussion of parameters of interest. In the cases where the condition $e\tau H/m^* c < 1$ is not fulfilled, expansion of the transport tensors in powers of H is not possible. Therefore the directional properties cannot be specified by a certain number of galvanomagnetic coefficients, and the problem is more complicated.

Experimentally, one commonly determines directional magnetoresistance coefficients $\Upsilon^{mnp}_{hkl}(H)$, operationally defined in (11.2). There are also directional Hall coefficients written as $R^{mnp}_{hkl}(H)$. It is now especially important to pay attention to the intermixing of contributions to E_{meas}

from the Hall field and from the magnetoresistance voltage. To illustrate, we shall write the general form of the resistivity tensor, making use only of the Kohler-Onsager reciprocal relations. We begin by writing the second rank tensor as a sum of symmetric and antisymmetric parts

$$\rho_{ik}(\mathbf{H}) = \rho^s_{ik}(\mathbf{H}) + \rho^a_{ik}(\mathbf{H}), \tag{11.7}$$

$$\rho^s_{ik}(\mathbf{H}) = \rho^s_{ki}(\mathbf{H}) = \tfrac{1}{2}\,[\rho^s_{ik}(\mathbf{H}) + \rho^s_{ki}(\mathbf{H})], \tag{11.8}$$

$$\rho^a_{ik}(\mathbf{H}) = -\,\rho^a_{ki}(\mathbf{H}) = \tfrac{1}{2}\,[\rho^a_{ik}(\mathbf{H}) - \rho^a_{ki}(\mathbf{H})].$$

Use of the Kohler-Onsager relations with (11.8) yields

$$\rho^s_{ik}(\mathbf{H}) = \rho^s_{ik}(-\,\mathbf{H}), \tag{11.9}$$

$$\rho^a_{ik}(\mathbf{H}) = -\,\rho^a_{ik}(-\,\mathbf{H}).$$

According to the last relation of (11.8), it is apparent that

$$\rho^a_{ik}(\mathbf{H}) = 0, \qquad i = k. \tag{11.10}$$

As a result of the preceding, it follows that there are six independent components in the resistivity tensor $\boldsymbol{\rho}(\mathbf{H})$; namely,

$$\boldsymbol{\rho}(\mathbf{H}) = \begin{pmatrix} \rho^s_{11}(\mathbf{H}) & \rho^s_{12}(\mathbf{H}) + \rho^a_{12}(\mathbf{H}) & \rho^s_{13}(\mathbf{H}) + \rho^a_{13}(\mathbf{H}) \\ \rho^s_{12}(\mathbf{H}) - \rho^a_{12}(\mathbf{H}) & \rho^s_{22}(\mathbf{H}) & \rho^s_{23}(\mathbf{H}) + \rho^a_{23}(\mathbf{H}) \\ \rho^s_{13}(\mathbf{H}) - \rho^a_{13}(\mathbf{H}) & \rho^s_{23}(\mathbf{H}) - \rho^a_{23}(\mathbf{H}) & \rho^s_{33}(\mathbf{H}) \end{pmatrix}. \tag{11.11}$$

Although the diagonal elements are even in H, the off-diagonal terms can include phenomena having both even and odd dependencies on H. Thus, unless highly symmetrical directions are chosen for **H**, **J**, and **E**, one may expect complications from these cross effects. The simplification of doing experiments which avoid the off-diagonal components is not useful, since standard measurements yield directly the elements of the *resistivity* tensor. To compare with theory, it is desirable to obtain the *conductivity*, or inverse tensor; and the inversion process requires a knowledge of all the components.

Because of the cross effects, it seems desirable to adopt the definitions of Hall and magnetoresistance coefficients suggested in Section 10*a*(3), namely, that the Hall phenomena are represented by the $\rho^a_{ik}(H)$ and the magnetoresistance phenomena by the $\rho^s_{ik}(H)$ of (11.11).[153b] The Hall and magnetoresistance fields are then given by

$$\mathbf{E}^a(\mathbf{H}) = \tfrac{1}{2}\,[\mathbf{E}(\mathbf{H}) - \mathbf{E}(-\,\mathbf{H})], \tag{11.12}$$

$$\mathbf{E}^s(\mathbf{H}) = \tfrac{1}{2}\,[\mathbf{E}(\mathbf{H}) + \mathbf{E}(-\,\mathbf{H})], \tag{11.13}$$

where $\mathbf{E}(\mathbf{H})$ is the electric field in the crystal when magnetic field $\mathbf{H}$ exists.

(*2*) *Hall phenomena*. The general Hall vector, $\mathbf{R}(\mathbf{H})$ of Eq. (8.21), is defined by

$$\mathbf{E}^a(\mathbf{H}) = \mathbf{R}(\mathbf{H}) \times \mathbf{J}. \tag{11.14}$$

The general Hall field is therefore perpendicular to $\mathbf{J}$, but not necessarily to $\mathbf{H}$,[63] and it is odd in $\mathbf{H}$.

(i) *Conventional Hall effect:* Many authors adopt the convention that the Hall field by definition is that field in the direction $\mathbf{H} \times \mathbf{J}$. We shall designate the quantity so defined by $\mathbf{E}_H$. It is sometimes known as the *transverse* Hall field, in distinction to the *longitudinal* Hall field, which will be discussed subsequently. The expression for the conventional Hall field may therefore be written

$$\mathbf{E}_H = \mathbf{E}^a(\mathbf{H}) \cdot \frac{\mathbf{H} \times \mathbf{J}}{|\mathbf{H} \times \mathbf{J}|}\,\frac{\mathbf{H} \times \mathbf{J}}{|\mathbf{H} \times \mathbf{J}|} = \frac{\mathbf{E}^a(\mathbf{H}) \cdot \mathbf{H} \times \mathbf{J}}{(\mathbf{H} \times \mathbf{J})^2}\,\mathbf{H} \times \mathbf{J}. \tag{11.15}$$

In this convention, the Hall field is expressed in terms of a Hall coefficient R_H as follows[153b,c]

$$\mathbf{E}_H = R_H\,\mathbf{H} \times \mathbf{J} \tag{11.16}$$

where the subscript H indicates a function of magnetic field. The weak-field limit is written as R_0, where

$$R_0 = \lim_{H \to 0} R_H. \tag{11.17}$$

[153b] This is the convention adhered to by Jan[43] and by Herring.[63]

[153c] Examples of the adoption of this convention in anisotropic solids for arbitrary magnetic field strengths are found in references 154–156.

[154] M. Shibuya, *Phys. Rev.* **95**, 1385 (1954).

From (11.12), (11.15), and (11.16) it follows that

$$R_{hkl}^{mnp}(H) = R_H = \tfrac{1}{2}[\mathbf{E}(\mathbf{H}) \cdot \mathbf{H} \times \mathbf{J} - \mathbf{E}(-\mathbf{H}) \cdot \mathbf{H} \times \mathbf{J}]/(\mathbf{H} \times \mathbf{J})^2. \qquad (11.18)$$

To avoid duplication of subscripts, we are writing the directional Hall coefficient R_H as $R_{hkl}^{mnp}(H)$. Since it is not a vector, there should be no confusion with $\mathbf{R}(\mathbf{H})$ in (11.14). The expression (11.18) above is identical with that given by Gold and Roth.[155]

(ii) *Longitudinal Hall effect:* The general relation (11.14) can, of course, specify a Hall field which is not collinear with $\mathbf{H} \times \mathbf{J}$. In particular, we may consider the direction $\mathbf{J} \times (\mathbf{H} \times \mathbf{J})$. This direction is parallel to $\mathbf{H}$ when $\mathbf{H}$ and $\mathbf{J}$ are perpendicular. The Hall field in question has been called the longitudinal Hall field by Grabner.[156a] Its magnitude is given by the relation

$$E_{\mathbf{J} \times (\mathbf{H} \times \mathbf{J})} = \frac{\mathbf{E}^a(\mathbf{H}) \cdot \mathbf{J} \times (\mathbf{H} \times \mathbf{J})}{|\mathbf{J} \times (\mathbf{H} \times \mathbf{J})|}. \qquad (11.18a)$$

The longitudinal Hall effect has been investigated both theoretically and experimentally for *n*-type germanium by Grabner.[156a] It vanishes for spherical energy surfaces or when the magnetic field is parallel to an axis of rotation of the crystal. It also approaches zero in the limit of infinitely strong magnetic fields.

(iii) *Longitudinal magnetic field Hall phenomena:* The Hall field expression, Eq. (11.14), can formally yield a nonzero field when $\mathbf{H}$ is parallel to $\mathbf{J}$, although such a situation is precluded for the conventional Hall field defined by Eq. (11.16). In most measurements the existence of a Hall voltage of this type is ruled out by considerations of symmetry. It has been pointed out, however, that such an effect, of third order in H or higher, can occur even in a cubic crystal if $\mathbf{J}$ is not along a direction of symmetry.[156b]

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[155] L. Gold and L. Roth, *Phys. Rev.* **107**, 358 (1957).

[156] W. M. Bullis, *Phys. Rev.* **109**, 292 (1958).

[156a] L. Grabner, *Phys. Rev.* **117**, 689 (1960).

[156b] See p. 84 of the article by García-Moliner.[428]

(*3*) *Magnetoresistance phenomena.* All galvanomagnetic effects which are even in the magnetic field can be accounted for symbolically by the following generalized magnetoresistivities:

$$ {}_{[\mathbf{E}]}\rho^{[\mathbf{H}]}_{[\mathbf{J}]}(H) \tag{11.19}$$

where only even powers of H are involved and where the crystallographic directions of **E**, **J**, and **H** are indicated. When $\mathbf{E} \parallel \mathbf{J}$, the above quantity represents the ordinary magnetoresistivity, designated as $\rho(H)$ in Eq. (11.2), and the advance subscript is omitted. For transverse directions of **E**, (11.19) will encompass resistivities leading to even-power "Hall" terms [Section 10*a*(1)] and to planar "Hall" phenomena [Section 10*a*(2)]. Thus

$${}_{[\mathbf{H}\times\mathbf{J}]}\rho^{[\mathbf{H}]}_{[\mathbf{J}]}(H), \quad \text{responsible for even-power "Hall" terms, } \mathbf{E} \text{ normal to } \mathbf{J} \text{ and to } \mathbf{H}, \tag{11.20}$$

$${}_{[(\mathbf{J}\times\mathbf{H})\times\mathbf{J}]}\rho^{[\mathbf{H}]}_{[\mathbf{J}]}(H), \quad \text{responsible for even-power terms via planar "Hall" effect, } \mathbf{E} \text{ normal to } \mathbf{J} \text{ and coplanar with } \mathbf{J} \text{ and } \mathbf{H}. \tag{11.21}$$

The resistivities in (11.19) are even in H and are given in the usual way by the double dot product of the symmetric part of the resistivity tensor with unit vectors in the specified directions. Thus

$$ {}_{[\mathbf{E}]}\rho^{[\mathbf{H}]}_{[\mathbf{J}]}(H) = \hat{\mathbf{E}} \cdot \boldsymbol{\rho}^s(\mathbf{H})\hat{\mathbf{J}} \tag{11.22}$$

where $\hat{\mathbf{E}}$ and $\hat{\mathbf{J}}$ are unit vectors in the direction of **E** and **J**, respectively. Hence, for the even-power "Hall" phenomena, we have

$$ {}_{[\mathbf{H}\times\mathbf{J}]}\rho^{[\mathbf{H}]}_{[\mathbf{J}]}(H) = \frac{(\mathbf{H}\times\mathbf{J})\cdot\boldsymbol{\rho}^s(\mathbf{H})\mathbf{J}}{|\mathbf{H}\times\mathbf{J}|\,|\mathbf{J}|} = \frac{\mathbf{H}\times\mathbf{J}\cdot\mathbf{E}^s(\mathbf{H})}{|\mathbf{H}\times\mathbf{J}|\,|\mathbf{J}|}, \tag{11.23}$$

$$ {}_{[(\mathbf{J}\times\mathbf{H})\times\mathbf{J}]}\rho^{[\mathbf{H}]}_{[\mathbf{J}]}(H) = \frac{(\mathbf{J}\times\mathbf{H})\times\mathbf{J}\cdot\boldsymbol{\rho}^s(\mathbf{H})\mathbf{J}}{|(\mathbf{J}\times\mathbf{H})\times\mathbf{J}|\,|\mathbf{J}|} = \frac{(\mathbf{J}\times\mathbf{H})\times\mathbf{J}\cdot\mathbf{E}^s(\mathbf{H})}{|\mathbf{J}\times\mathbf{H}|\,J^2} \tag{11.24}$$

where the $\mathbf{E}^s(\mathbf{H})$ are given by (11.13).

The fields for these even-power "Hall" effects are given by

$$\mathbf{E}_H^s = \frac{[\mathbf{E}^e(\mathbf{H}) \cdot \mathbf{H} \times \mathbf{J}]\mathbf{H} \times \mathbf{J}}{(\mathbf{H} \times \mathbf{J})^2}, \tag{11.25}$$

$$\mathbf{E}_H^p = \frac{[\mathbf{E}^e(\mathbf{H}) \cdot (\mathbf{J} \times \mathbf{H}) \times \mathbf{J}][(\mathbf{J} \times \mathbf{H}) \times \mathbf{J}]}{(\mathbf{J} \times \mathbf{H})^2 J^2} \tag{11.26}$$

The voltage $\mathbf{E}_H^p$ is sometimes expressed in terms of a planar Hall coefficient, which we shall write as $\mathscr{R}_H^p$. Such a representation is possible if the system has sufficient symmetry so that the contributions to $\mathbf{E}_H^p$ from $\mathbf{J}$ arise from a single type of galvanomagnetic coefficient [Section 10*a*(2)]. This is true, for example, in a cubic crystal when $\mathbf{E}$ and $\mathbf{J}$ are along cube axes, or in a trigonal system when the plane of $\mathbf{E}$, $\mathbf{J}$, and $\mathbf{H}$ is normal to the threefold axis. In such cases, we have from Eq. (10.1), to first order in H^2,

$$E_2 = 2\rho_{2121}^0 JH^2 \sin\varphi \cos\varphi = \mathscr{R}_0^p JH^2 \sin\varphi \cos\varphi. \tag{11.27}$$

Thus, for systems of appropriate symmetry, we may write

$$\mathbf{E}_H^p = \mathscr{R}_H^p [(\mathbf{J} \cdot \mathbf{H})/J^2] [(\mathbf{J} \times \mathbf{H}) \times \mathbf{J}]. \tag{11.28}$$

From (11.26) to (11.28) it follows that

$$\mathscr{R}_H^p = [\mathbf{E}^e(\mathbf{H}) \cdot (\mathbf{J} \times \mathbf{H}) \times \mathbf{J}]/(\mathbf{J} \times \mathbf{H})^2 (\mathbf{J} \cdot \mathbf{H}). \tag{11.29}$$

This expression for the planar "Hall" coefficient is similar to that given by Bullis,[156] except that our definition explicitly excludes any contributions from $\mathbf{E}^o(\mathbf{H})$.

If the vectors $\mathbf{E}$, $\mathbf{J}$, and $\mathbf{H}$ are designated relative to an orthogonal coordinate system, then all of the coefficients discussed in this section can be written in terms of components of the resistivity tensor (11.11). The expressions can be quite complex if the fields and the current are not in directions of high symmetry, as will be noted from the explicit results presented in a number of the references cited. In general, $\rho_{xy}(H) \neq -\rho_{yx}(H)$ because of nonzero symmetric parts in the off-diagonal elements. Also, even if the measurement directions are chosen so as that the observed effects can be described by a two-dimensional $\rho_{ik}(\mathbf{H})$, the transformation to $\sigma_{ik}(\mathbf{H})$ whose components are desired for a comparison with theory, requires a knowledge of the other components of the $\rho_{ik}(H)$ tensor. Thus the simple results obtained in the isotropic case — e.g., Eqs. (9.12) and (9.14) — represent a tremendous simplification from what may be encountered in an anisotropic system.

(*4*) *Illustrative examples.* By measuring a number of parameters from among those given in (11.14) or (11.18), (11.19) — or in some cases, (11.29) — it is possible to determine the components of the resistivity tensor $\boldsymbol{\rho}(\mathbf{H})$ for specified directions of $\mathbf{H}$. Through inversion of the resistivity tensor the components of $\boldsymbol{\sigma}(\mathbf{H})$ can be obtained for comparison with theory.

Most of the effort in this direction has been applied to cubic crystals, in particular, the conduction bands of silicon and germanium, where the many-valley approximation to the band structure is applicable. Since mathematically exact evaluations of the transport coefficients for arbitrary values of H are possible only for quadratic energy surfaces, treatment of anisotropy by the ellipsoid model (discussed in detail in Section 23) has an important advantage.

The method, using ellipsoids of revolution to express $\varepsilon(\mathbf{k})$, was applied by Abeles and Meiboom[157] and by Shibuya[154] to germanium and silicon. Explicit expressions are given for Hall and magnetoresistance coefficients, in terms of the effective mass ratios which characterize the ellipsoids, for several crystallographic directions of $\mathbf{J}$ and $\mathbf{H}$. A relaxation time proportional to $\varepsilon^{-1/2}$ was used, corresponding to scattering by acoustic phonons. The expressions are easily evaluated in the weak and in the strong magnetic field limits.

For intermediate field strengths, the formulae for the $\varepsilon^{-1/2}$ dependence of $\boldsymbol{\rho}$, although tractable, are rather laborious to evaluate. Results are greatly simplified in the constant-τ treatment, which has been carried out by Gold and Roth.[155,158] Then the magnetic-field dependence enters directly through the term $\omega\tau$, rather than via more complicated functions of $\omega\tau$. It is of particular interest to note the large changes in the Hall coefficients R_H for the model representative of n-type germanium at intermediate field strengths as $\mathbf{J}$ and $\mathbf{H}$ are moved off the cube axes.

The experimental arrangements necessary to determine the components of $\boldsymbol{\rho}(\mathbf{H})$ for an arbitrary direction of $\mathbf{H}$ are discussed by Broudy and

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[157] B. Abeles and S. Meiboom, *Phys. Rev.* **95**, 31 (1954).

[158] L. Gold and L. Roth, *Phys. Rev.* **103**, 61 (1956).

Venables.[159] The authors point out the necessity for redundancy as a check on the homogeneity of the sample. In the stronger magnetic field regions, the effects from inhomogeneities can be pronounced. General expressions are worked out, using spheroidal models representative of the conduction bands of germanium and silicon, for the components of the conductivity tensor.

Experimental data on Hall coefficient and magnetoresistance at room temperature in *n*- and *p*-type germanium as a function of magnetic field are given by Della Pergola and Sette.[160] Results on the Hall coefficient in *n*-type germanium at 77°K have been presented by Bullis and Krag.[161] These authors define a theoretical R_H^{th} in terms of $\mathbf{E} \cdot \mathbf{J} \times \mathbf{H}$, where $\mathbf{E}$ is the total electric field in the crystal, rather than the antisymmetric field used in (11.18). Contributions to the R_H^{th} defined above from the symmetrical parts of the resistivity tensor are subsequently removed. The magnitude of the cross effects at intermediate magnetic field strengths can be substantial for the less highly symmetric directions, as is seen in Fig. 9.

Much more extensive data are available in subsequent articles by Bullis[162] on *n*-type germanium and by Krag[163] on *n*-type silicon. Magnetoresistance, Hall coefficients, and planar "Hall" coefficients were measured at 77° and at 300°K for a number of magnitudes and orientations of **H**. A detailed investigation of the anisotropy of the Hall coefficient in germanium containing 5×10^{14} donors/cm^3 was done by Miyazawa and Maeda.[163a] A number of orientations were studied, and data were taken over a range of temperatures and magnetic field strengths.

In the valence bands of germanium and silicon, the anisotropy is much less pronounced than in the conduction bands. Experimental data on the directional properties of the galvanomagnetic effects as a function of magnetic field for *p*-type germanium and silicon are meager in the

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[159] R. M. Broudy and J. D. Venables, *Phys. Rev.* **105**, 1757 (1957); **103**, 1129 (1956).

[160] G. C. Della Pergola and D. Sette, *Nuovo cimento* **5**, 1670 (1957).

[161] W. M. Bullis and W. E. Krag, *Phys. Rev.* **101**, 580 (1956).

[162] W. M. Bullis, *Phys. Rev.* **109**, 292 (1958).

[163] W. E. Krag, *Phys. Rev.* **118**, 435 (1960).

[163a] H. Miyazawa and H. Maeda, *J. Phys. Soc. Japan* **15**, 1924 (1960); "Proc. of the Intern. Conf. on Semiconductor Physics, Prague, 1960," p. 169, Publishing House Czech. Acad. Sci., Prague, 1961.

literature. The orientations used by Della Pergola and Sette in germanium show a negligible effect at room temperature.[160] Results of investigations by Beer and Willardson[56,164] reveal fine structure in the Hall coefficient

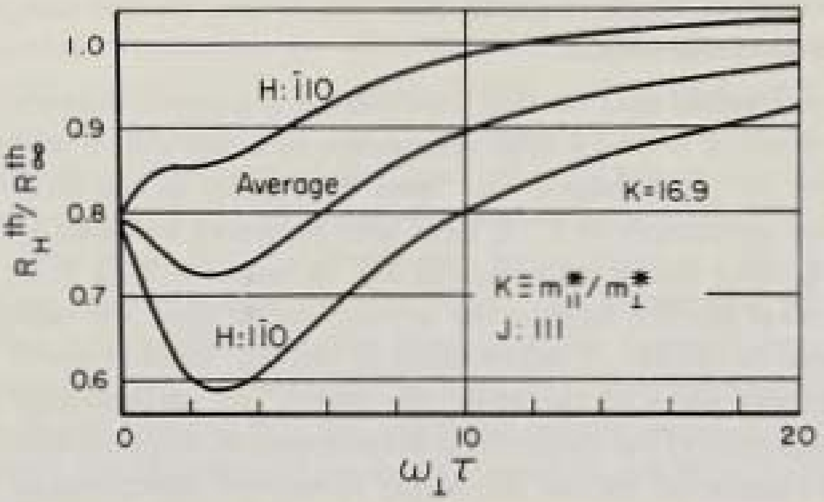

FIG. 9. Relative "theoretical" Hall coefficient (see text) as a function of $\omega_\perp\tau$ ($\equiv eH\tau/m_\perp^* c$) for **J** in the [111] direction (after Bullis and Krag[161]). The quantity $m_\parallel^*$ is the longitudinal mass (that associated with the axis of revolution of the energy spheroid), and $m_\perp^*$ is the transverse mass.

of *p*-type germanium and silicon at 77 °K. Data on germanium for **H**[100] and **H**[1$\bar{1}$1] suggest an enhancement of the structure in the latter case. This enhancement of structure for $R_H^{[111]}$ was also observed by Miyazawa[164a] in his studies of the anisotropy of the Hall effect in *p*-type germanium, using specimens containing acceptor densities ranging from 2×10^{13} to 1×10^{16} cm^{-3}. Magnetoresistance studies on *p*-type silicon carried out by Long,[125] show the coefficient $\Upsilon_{110}^{1\bar{1}0}$ to be several per cent higher than Υ_{110}^{001} at 77 °K; at room temperature, however, the difference is over 30%. Considerable augmentation also occurs in the ratio of longitudinal and transverse coefficients at the higher temperature. These results suggest an anisotropy becoming more pronounced with increase in temperature. At the higher temperature, however, transport may be complicated by contributions from the split-off band (see Section 17*b*).

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[164] A. C. Beer, *Phys. and Chem. Solids* 8, 507 (1959).

[164a] H. Miyazawa, *in* "Proc. of the Conf. on the Physics of Semiconductors, Exeter, July, 1962," p. 636. The Institute of Physics and the Physical Society, London, 1962.

The band structure for p-type germanium and silicon cannot, of course, be approximated by the ellipsoidal model since the band edge is at $\mathbf{k} = 0$. Instead, one must deal with degenerate bands, where one or more is described as a "warped sphere" (Chapter VII).

V.

Transport Coefficients for Isothermal Solids Assuming a Single Parabolic Energy Band and the Relaxation Time Approximation

This section is devoted to transport by a single band having spherical energy surfaces: first for exact quantum (Fermi-Dirac) statistics, and second for the classical (Maxwell-Boltzmann) approximation. In the latter case, especially, results are obtained for a number of different dependencies of the relaxation time on energy, including a mixed scattering process. The limiting cases of weak and strong magnetic fields are considered in some detail; and for certain scattering mechanisms, solutions are given for arbitrary values of magnetic field strength.

12. GENERAL CONSIDERATIONS — EXACT STATISTICS

a. Expressions for Conductivity Coefficients

It is instructive to express the conductivity coefficients in terms of the charge carrier density n. This parameter is given by an integration over energy of the product of the density of states by the Fermi-Dirac distribution function, which expresses the probability of occupation of a given state of energy, designated by ε. Thus, using (5.0a) we may write

$$n = \int_0^\infty n(\varepsilon) f_0(\varepsilon)\, d\varepsilon = \frac{1}{4\pi^3} \int_{\varepsilon=0}^{\varepsilon=\infty} f_0(\varepsilon)\, d^3k(\varepsilon). \tag{12.1}$$

The Fermi-Dirac distribution function is given by

$$f_0(\varepsilon) = \frac{1}{e^{\varepsilon-\eta}+1} \tag{12.2}$$

where x is the reduced energy of the charge carrier and η is the reduced Fermi energy[165]:

$$x \equiv \varepsilon/kT; \qquad \eta \equiv \zeta/kT. \tag{12.3}$$

For spherical energy surfaces

$$d^3k = 4\pi k^2\, dk \tag{12.4}$$

and

$$\varepsilon(k) = [\hbar^2/2m^*]\,[k_x^2 + k_y^2 + k_z^2] = \hbar^2 k^2/2m^*. \tag{12.5}$$

The expression for the carrier density can therefore be put in the form

$$n = \frac{1}{2\pi^2}\left(\frac{2m^*\,kT}{\hbar^2}\right)^{3/2} F_{1/2}(\eta). \tag{12.6}$$

The quantity $F_{1/2}(\eta)$ is the Fermi-Dirac function of order 1/2, defined generally as

$$F_k(\eta) \equiv \int_0^\infty \frac{x^k}{e^{x-\eta}+1}\, dx. \tag{12.7}$$

The characteristics of the Fermi-Dirac functions have been investigated by McDougall and Stoner[166] and values are tabulated for $k = -1/2$, 1/2, and 3/2. Results were extended by Beer *et al.*[167] to include $k = 5/2$, 7/2, 9/2, and 11/2. The behavior of the functions for even indices was studied by Rhodes[168] and he presents values for $k = 1, 2, 3$, and 4. Relationships of the Fermi-Dirac integrals to other functions have been investigated by Dingle[169] and a number of expansions were developed.

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[165] For sake of custom, it is desirable to use the same symbol for the Boltzmann constant in (12.3) as is used for the magnitude of the wave vector. It will be apparent from the formulae which designation is meant. Furthermore, the Boltzmann constant usually occurs in conjunction with T. The exceptions will be noted.

[166] J. McDougall and E. C. Stoner, *Phil. Trans. Roy. Soc. London* **A237**, 67 (1938).

[167] A. C. Beer, M. N. Chase, and P. F. Choquard, *Helv. Phys. Acta* **28**, 529 (1955).

[168] P. Rhodes, *Proc. Roy. Soc.* **A204**, 396 (1950).

[169] R. B. Dingle, *Appl. Sci. Research* **B6**, 225 (1957).

These are sufficiently complete so that integrals of all orders can be calculated without recourse to numerical integration. In his development, Dingle introduces the function $\mathcal{F}_k(\eta)$, where $\mathcal{F}_k(\eta)$ is defined as

$$\mathcal{F}_k(\eta) = F_k(\eta)/k!. \tag{12.8}$$

This representation has a number of mathematical advantages in connection with interpolation, classical limit as $\eta \ll 0$, etc. Also, unlike $F_k(\eta)$, the function $\mathcal{F}_k(\eta)$ exists even for negative integer indices. Tabulated values of $\mathcal{F}_k(\eta)$ are given by Dingle for integer indices from -1 to 4. A further tabulation of $F_k(\eta)$ by Madelung[170] goes through $F_7(\eta)$.

With use of relation (12.6), the expression (7.15) for the conductivity coefficient $\sigma_{xx}(H)$ can be put in the form

$$\sigma_{xx}(H) = \frac{ne^2}{m^*}\left[-\frac{2}{3}\frac{1}{F_{1/2}(\eta)}\int_0^\infty \frac{\tau}{1+\omega^2\tau^2}\,x^{3/2}\,\frac{\partial f_0}{\partial x}\,dx\right], \qquad H \equiv H_z. \tag{12.9}$$

To gain an insight into meaning of the expression in brackets, we evaluate $\sigma'_{xx}(H)$ for the case of constant τ, i.e., τ independent of ε. After an integration by parts, one obtains

$$\sigma_{xx}(H) = \frac{n\,e^2}{m^*}\left[\frac{\tau}{1+\omega^2\tau^2}\right], \qquad \tau = \text{const.} \tag{12.10}$$

In the general case, we may regard the bracketed factor in (12.9) as an average of $\tau/(1+\omega^2\tau^2)$ over the Fermi-Dirac distribution. Thus, for a general function of energy, $q(\varepsilon/kT)$, we define

$$\langle q\rangle_{\text{F-D}} \equiv -\frac{2}{3}\frac{1}{F_{1/2}(\eta)}\int_0^\infty q(x)\,x^{3/2}\,\frac{\partial f_0}{\partial x}\,dx. \tag{12.11}$$

The concept of the average is also brought out by the observation that

$$F_{1/2}(\eta) = -\frac{2}{3}\int_0^\infty x^{3/2}\,\frac{\partial f_0}{\partial x}\,dx. \tag{12.11a}$$

The conductivity coefficients can now be written

$$\sigma_{xx}(H) = \frac{ne^2}{m^*}\left\langle\frac{\tau}{1+\omega^2\tau^2}\right\rangle_{\text{F-D}}, \qquad H \equiv H_z. \tag{12.12}$$

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[170] O. Madelung, pp. 53–62 of the 1957 "Handbuch."[122]

$$\sigma_{xy}(H) = -\frac{ne^2\,\omega}{m^*}\left\langle\frac{\tau^2}{1+\omega^2\,\tau^2}\right\rangle_{\mathrm{F-D}}, \qquad (12.13)$$

$\omega = eH/m^*\,c$, and $e > 0$ for electrons.

In principle, $\sigma_{xx}(H)$ and $\sigma_{xy}(H)$ can be evaluated for arbitrary values of H if $\tau(\varepsilon)$ is known. However, numerical integrations are necessary in most cases, and calculations are laborious even for simple energy dependencies such as $\tau \sim \varepsilon^n$. The problem is greatly simplified if one is interested only in the behavior in the limiting cases of weak and strong magnetic fields.

In dealing with transport problems, several investigators have found it advantageous to introduce *complex* gradients.[171, 172] In fact, it is shown by Dingle that it is possible to introduce simultaneously two complex operators, one of spatial rotation around the z-axis to account for the two orthogonal components of the electrical and thermal gradients in the xy-plane, and another as a measure of phase lag for treating effects in high-frequency electric fields. This technique has been used by Moore[173] to present in a concise manner general formulae describing both electrical and thermal transport for the gradient vectors in the xy-plane with $\mathbf{H}$ along the z-axis. Results are also given for the two-band model, i.e., where both electrons and holes contribute to the transport process. Explicit expressions are provided for most of the electrical and thermal effects with and without magnetic fields.

b. Limiting Cases of Weak and of Strong Magnetic Fields

For weak fields the denominators in (12.12) and (12.13) are usually expanded in powers of ω.[174] The results, up to powers of H^2, are

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[171] E. H. Sondheimer, *Phys. Rev.* **80**, 401 (1950).

[172] R. B. Dingle, *Physica* **22**, 701 (1956).

[173] E. J. Moore, *Australian J. Phys.* **11**, 235 (1958).

[174] Although such expansions are made to render evaluation of the integrals less laborious, certain mathematical difficulties can arise because of the fact that τ and powers thereof are averaged over energy from $\varepsilon = 0$ to $\varepsilon = \infty$. Thus if τ is approximated by an ε^λ dependence and $\lambda < 0$, it is apparent that in the neighborhood of $\varepsilon = 0$, the term $\omega\tau$ can actually become very large for any nonzero value of ω, however small. The expansion is thus invalid in this region. If λ is sufficiently negative the contributions to the integrand from this region can actually cause the integral to diverge. From a physical standpoint, infinite relaxation times are not realistic; additional scattering processes come in for the low-energy electrons.[175]

$$\sigma_{xx}(H) \cong \frac{ne^2}{m^*}\{\langle\tau\rangle_{\mathrm{F-D}} - \langle\tau^3\rangle_{\mathrm{F-D}}\,\omega^2\}, \qquad \omega\tau \ll 1, \tag{12.14}$$

$$\sigma_{xy}(H) \cong -\frac{ne^2}{m^*}\langle\tau^2\rangle_{\mathrm{F-D}}\,\omega. \tag{12.15}$$

For strong fields, the denominator reduces to $\omega^2\tau^2$, and the result is

$$\sigma_{xx}(H) \approx \frac{ne^2}{m^*}\left\langle\frac{1}{\tau}\right\rangle_{\mathrm{F-D}}\frac{1}{\omega^2}, \qquad \omega\tau \gg 1 \tag{12.16}$$

$$\sigma_{xy}(H) \approx -\frac{ne^2}{m^*}\frac{1}{\omega} \approx -\frac{nec}{H}. \tag{12.17}$$

In connection with the factor $\langle 1/\tau\rangle_{\mathrm{F-D}}$, mathematical difficulties can arise for $\tau \sim \varepsilon^{\lambda}$, where λ is positive.[175]

It is seen from the preceding development that at weak fields σ_{xx} dominates σ_{xy}, but at high fields the reverse occurs.

An especially interesting point is that the asymptotic form of $\sigma_{xy}(H)$ is *independent of the relaxation time.* Although this fact is shown here specifically for spherical energy surfaces, it is readily established for a general quadratic relation, $\varepsilon(k)$, that $\sigma^{a}_{xy}(H)$ — the part of $\sigma_{xy}(H)$ which is odd in H — behaves as does (12.17).[65] This independence of $\sigma^{a}_{xy}(H)$ in the high-magnetic-field limit on the nature of the scattering mechanism has in fact been shown to hold quite generally[176] — for all scattering processes represented by the general collision integral, Eq. (5.8), with no restriction on the shape of the energy surfaces other than that the hodograph in the magnetic field (see Section 6*b*) be a closed curve.[177] A similar result is obtained by Lifshitz from his quantum-mechanical treatment of conductivity in a magnetic field.[42]

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[175] The argument is similar to that advanced for the weak-field case[174] except that here the divergences arise for sufficiently large positive values of λ.

[176] J. A. Swanson, *Phys. Rev.* **99**, 1799 (1955).

[177] I. Lifshitz, M. Azbel' and M. Kaganov, *J. Exptl. Theoret. Phys. U.S.S.R.* **30**, 220 (1956) [translation: *Soviet Phys.-JETP* **3**, 143 (1956)].

For certain applications, the relaxation time can be approximated by the simple power-law dependence on energy[178]:

$$\tau = a\varepsilon^{\lambda} = \tau_0\, x^{\lambda}. \tag{12.18}$$

In such cases the averages in (12.14) to (12.16) can be expressed in terms of the Fermi-Dirac functions by means of an integration by parts to yield

$$\sigma_{xx}(H) \cong \frac{2ne^2}{3m^* F_{1/2}(\eta)} \{\tau_0(\lambda + \tfrac{3}{2})\, F_{\lambda+(1/2)}(\eta) - \omega^2\, \tau_0{}^3(3\lambda + \tfrac{3}{2})\, F_{3\lambda+(1/2)}(\eta)\},$$

$$\omega\tau \ll 1, \qquad (12.19)$$

$$\sigma_{xy}(H) \cong -\frac{2}{3}\frac{ne^2}{m^* F_{1/2}(\eta)}\, \omega\tau_0{}^2(2\lambda + \tfrac{3}{2})F_{2\lambda+1/2}(\eta). \tag{12.20}$$

$$\sigma_{xx}(H) \approx \frac{2}{3}\frac{ne^2}{m^* F_{1/2}(\eta)}\frac{1}{\omega^2}\frac{1}{\tau_0}(\tfrac{3}{2} - \lambda)F_{(1/2)-\lambda}(\eta), \qquad \omega\tau \gg 1. \tag{12.21}$$

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[178] This approximation, while helpful in certain cases, must be used with caution. Negative values of λ, which are representative of many high-purity semiconductors, yield values of τ which increase rapidly as ε approaches zero. If low-energy charge carriers contribute too strongly to the integrals of the type $\langle\tau^r\rangle_{F-D}$, not only will unrealistic contributions result, but the integrals may actually diverge. This situation has been pointed out by a number of authors,[179] and is discussed in detail by Brooks.[180] Two aspects are involved in the divergence of the integrals: (1) The inadequacy of (12.18) to represent τ satisfactorily over the range of integration $0 \leqslant \varepsilon < \infty$ and (2) the failure of the mathematical expansion of $[1 + \omega^2\tau^2]^{-1}$ over part of this region. In regard to (1), the relaxation time for low-energy carriers is actually limited by other scattering processes such as those from neutral impurities, dislocations, and ionized impurities. In semiconductors, the latter process, with $\tau \underset{\sim}{\propto} \varepsilon^{3/2}$, is usually the important one. The difficulty introduced as a result of (1) can be corrected by taking account of composite or mixed scattering — this precludes use of the simple expression (12.18) — or by applying a suitable cutoff to the low energy limit of the integral involving (12.18).[180]

The difficulty in item (2) can, of course, be overcome by avoiding the expansion and obtaining numerical evaluations. Where classical statistics are applicable, the integrals can sometimes be evaluated in terms of tabulated functions (see Section 15).

Brooks points out that the factor determining which situation predominates is whether $\omega^2\tau^2$ is smaller or larger than unity at that energy for which τ reaches its maximum when the scattering process effective at low energies is taken into account.

[179] See, for example, p. 253 of reference 44.

[180] See pp. 130–133 of reference 33; and, in particular, the article by Benedek *et al.*[329]

For the case of scattering in semiconductors by thermal lattice vibrations through acoustic modes ($\lambda = -\frac{1}{2}$), and scattering by ionized impurities ($\lambda = \frac{3}{2}$), the weak-field expressions reduce to those given by Madelung.[170] For $\lambda = -\frac{1}{2}$, the product $(3\lambda + \frac{3}{2})F_{3\lambda+1/2}(\eta)$ becomes indeterminate. It can be shown that the limit is given by Dingle's function $\mathscr{F}_{-1}(\eta)$, which is identical with $(1 + e^{-\eta})^{-1}$. One can, of course, insert the specific value of λ in (12.18) and evaluate directly the second term of (12.14) by means of (12.11). Since the integrand contains no powers of x, there is no partial integration and one obtains directly $\int_0^\infty df_0(\varepsilon)$. For a scattering process approximated by (12.18) with $\lambda < -\frac{3}{2}$, difficulties can arise with expansions (12.19) and (12.20).[174, 178] A similar situation is apparent with (12.21) for $\lambda > \frac{3}{2}$.[175]

Madelung also gives the form of the integrals when mixed scattering by thermal lattice vibrations and ionized impurities is considered. For this case the relaxation time is approximated by

$$1/\tau = 1/\tau_L + 1/\tau_I = (\tau_L{}^0)^{-1}\, x^{1/2} + (\tau_I{}^0)^{-1}\, x^{-3/2}. \tag{12.22}$$

In the present case of exact statistics, the integrals must be evaluated numerically.

c. Conductivity Mobility

We shall define a general conductivity mobility, μ, by the relation[180a]

$$\sigma(H) = -ne\mu(\eta), \qquad e > 0 \text{ for electrons.} \tag{12.23}$$

The quantity defined above is a function of the reduced Fermi energy η (measured from the bottom of the conduction band for electrons) and of the magnetic field intensity H. The quantity η is often referred to as the degeneracy parameter. The value of the conductivity mobility for

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[180a] The mobility, which is the charge carrier velocity (suitably averaged) per unit electric field, should strictly carry a negative sign for electrons and a positive sign for holes. This is the convention adopted in (12.23). There is considerable tendency in the literature, however, to regard μ as a positive quantity and to take care of the polarity of the charge carriers explicitly by means of the algebraic signs preceding each term. This latter convention will be adopted here whenever μ is supplied with a subscript to denote a particular band, especially the electron or hole band.

vanishing magnetic fields is denoted by $\mu_0(\eta)$. In the limit of classical statistics, $\eta \ll 0$, the mobility becomes independent of η, and is written as μ and μ_0, respectively.

By using (9.13), (12.12), and (12.13) we find that

$$\mu(\eta) = -\frac{e}{m^*}\left\{\left\langle\frac{\tau}{1+\omega^2\tau^2}\right\rangle_{\mathrm{F-D}} + \omega^2\left\langle\frac{\tau^2}{1+\omega^2\tau^2}\right\rangle^2_{\mathrm{F-D}} \Bigg/ \left\langle\frac{\tau}{1+\omega^2\tau^2}\right\rangle_{\mathrm{F-D}}\right\}. \tag{12.24}$$

Unless specified otherwise, it is the zero-magnetic-field mobility which is usually quoted in the literature. Thus (12.24) simplifies greatly to give

$$\mu_0(\eta) = -(e/m^*)\,\langle\tau\rangle_{\mathrm{F-D}}, \qquad e > 0 \text{ for electrons.} \tag{12.25}$$

d. Hall Coefficient, Hall Mobility, and Magnetoresistance

The expressions for the Hall coefficient, conductivity, and Hall angle in isotropic media with $\mathbf{H}$ $(0, 0, H)$ and the boundary conditions $J_y = \partial T/\partial x = \partial T/\partial y = 0$ and $E_z = J_z = \partial T/\partial z = 0$ were given in (9.12) to (9.17), namely,

$$R_H = \rho_{yx}(H)/H = \sigma_{xy}(H)/H\,[\sigma_{xx}^2(H) + \sigma_{xy}^2(H)], \tag{12.26}$$

$$\sigma(H) = 1/\rho_{xx}(H) = [\sigma_{xx}^2(H) + \sigma_{xy}^2(H)]/\sigma_{xx}(H), \tag{12.27}$$

$$E_y/E_x \equiv \tan\theta = \sigma_{xy}(H)/\sigma_{xx}(H) = R_H\,\sigma(H)H. \tag{12.28}$$

It is not necessary to indicate antisymmetric parts of the ρ_{yx} and the ρ_{xy} (see Section 11*b*) since in isotropic media there are no symmetric contributions to the off-diagonal elements.

It is customary, especially in the weak-magnetic-field case, to introduce a *Hall mobility*,[181] defined by means of (12.28). Thus

$$\mu^H H/c = \tan\theta, \tag{12.29}$$

$$\cong \theta, \qquad \theta \ll 1. \tag{12.29a}$$

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[181] See, for example, p. 209 of reference 32.

We shall define this mobility parameter in a general manner for arbitrary magnetic fields and degree of degeneracy[182]:

$$\mu^H(\eta) \equiv R_H \sigma(H) c. \tag{12.30}$$

A useful relationship follows from (12.28) and (12.29), namely,

$$\begin{aligned} \sigma_{xy}(H) &= \sigma_{xx}(H) \tan\theta = (\mu^H H/c)\sigma_{xx}(H) \qquad\qquad (12.30\mathrm{a}) \\ &= R_H \sigma(H) H \sigma_{xx}(H) \qquad (\text{Gaussian units, } H \equiv H_z). \end{aligned}$$

The Hall mobility in the limit of vanishing magnetic field is designated by

$$\mu_0^H(\eta) \equiv R_0 \sigma_0 c. \tag{12.31}$$

In the case of classical statistics, the notation is μ^H and μ_0^H, respectively.

The transverse magnetoresistance, Eq. (9.14), in an isotropic system is given by

$$\frac{\Delta\rho}{\rho_0} \equiv \frac{\rho(H)}{\rho_0} - 1 = \frac{\sigma_0}{\sigma(H)} - 1 = \frac{\sigma_{xx}(H)\sigma_{xx}(0) - \sigma_{xx}^2(H) - \sigma_{xy}^2(H)}{\sigma_{xx}^2(H) + \sigma_{xy}^2(H)}. \tag{12.32}$$

The Corbino magnetoresistance (see Section 10*b*) is measured under the conditions that the transverse electric field, rather than the transverse electric current, be zero. Accordingly, we have

$$(\Delta\rho/\rho_0)_{E_y=0} = \frac{\sigma_{xx}(0)}{\sigma_{xx}(H)} - 1. \tag{12.33}$$

By means of relations (12.28), the Corbino magnetoresistance can be written in terms of the ordinary transverse magnetoresistance and the Hall angle, thus

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[182] It is important to keep in mind that the Hall mobility — certainly, as the term is commonly used — is indicative of the electronic processes in a given band. If multiband contributions to the transport occur, then there is associated with *each* band a Hall mobility — as there is also a conductivity mobility. If R is the *measured* Hall coefficient of the material, then (12.30) and (12.31) cannot be used unless single band conduction occurs. To use these expressions for multiband transport, requires that we determine R_H or R_0 *for the band in question*. This situation has caused considerable confusion in the literature. In particular, formulae such as (12.31) have been used in cases where intrinsic conduction is significant (both electrons and holes present) and where there are degenerate valence bands (several types of holes having different effective masses and mobilities).

$$(\Delta\rho/\rho_0)_{E_y=0} = (1+\tan^2\theta)(\Delta\rho/\rho_0) + \tan^2\theta \tag{12.34}$$

$$= \Delta\rho/\rho_0 + [\rho(H)/\rho_0]\tan^2\theta \tag{12.35}$$

$$= \Delta\rho/\rho_0 + R_H{}^2\,\sigma(H)\sigma(0)H^2 \tag{12.36}$$

$$= \Delta\rho/\rho_0 + \frac{\mu_0(\eta)}{\mu(\eta)}\left(\frac{\mu^H(\eta)H}{c}\right)^2. \tag{12.37}$$

The relationship (12.34) has been given by Madelung.[148]

Although explicit expressions can be given for all of the above coefficients by means of (12.12) and (12.13), we shall discuss only the results in the limits of low and high magnetic field strengths. For weak magnetic fields, $\omega\tau \ll 1$, one obtains

$$R_0 = \lim_{H\to 0}\left[\frac{\sigma_{xy}(H)}{H}\right]\cdot\frac{1}{\sigma_{xx}{}^2(0)} = -\frac{1}{nec}\,\frac{\langle\tau^2\rangle_{\mathrm{F-D}}}{\langle\tau\rangle^2_{\mathrm{F-D}}}, \qquad e>0 \text{ for electrons,} \tag{12.38}$$

$$\mu_0{}^H(\eta) = -\frac{e}{m^*}\,\frac{\langle\tau^2\rangle_{\mathrm{F-D}}}{\langle\tau\rangle_{\mathrm{F-D}}} \tag{12.39}$$

$$\frac{\mu_0{}^H(\eta)}{\mu_0(\eta)} = \frac{\langle\tau^2\rangle_{\mathrm{F-D}}}{\langle\tau\rangle^2_{\mathrm{F-D}}}, \tag{12.40}$$

$$\frac{\Delta\rho}{\rho_0} = \frac{\langle\tau^3\rangle_{\mathrm{F-D}}\,\langle\tau\rangle_{\mathrm{F-D}} - \langle\tau^2\rangle^2_{\mathrm{F-D}}}{\langle\tau\rangle^2_{\mathrm{F-D}}}\,\omega^2, \qquad \omega\tau \ll 1, \tag{12.41}$$

$$\left(\frac{\Delta\rho}{\rho_0}\right)_{E_y=0} = \omega^2\,\frac{\langle\tau^3\rangle_{\mathrm{F-D}}}{\langle\tau\rangle_{\mathrm{F-D}}}, \qquad \omega\tau \ll 1. \tag{12.41a}$$

Thus, in the weak-magnetic-field case,

$$(\Delta\rho/\rho_0)_{E_y=0} = \Delta\rho/\rho_0 + \omega^2\,\langle\tau^2\rangle^2_{\mathrm{F-D}}/\langle\tau\rangle^2_{\mathrm{F-D}} \tag{12.42}$$

$$= \Delta\rho/\rho_0 + (\mu_0{}^H\,H/c)^2. \tag{12.43}$$

The above expression is obviously the weak-field version of (12.37). In view of (12.25), the condition $\omega\tau \ll 1$ is, for practical purposes, equivalent to

$$\mu_0\,H/c \ll 1 \tag{12.44}$$

where the Gaussian system of units is implied. If laboratory units are used where μ is in cm^2/volt-sec and H in gauss, then the condition is

$$\mu_0 H/10^8 \ll 1. \tag{12.45}$$

A significant point, apparent from the preceding, is that the conductivity mobility μ_0 involves an average of τ, the Hall mobility $\mu_0{}^H$ includes averages of τ^2, and the transverse magnetoresistance coefficients include averages of τ^3.

It is also possible to define a magnetoresistance mobility, inasmuch as the dimensions of $\Delta\rho/\rho H^2$ are those of a mobility squared. However, the factor of proportionality is somewhat arbitrary. It is not possible to adopt the same convention as that used for $\mu_0{}^H$, which in the constant-τ approximation reduces to μ_0. The reason is that the magnetoresistance vanishes for constant τ in our isotropic model. It is necessary, therefore, to base the definition on a model where τ is a simple function of energy — as, for example, the $\varepsilon^{-1/2}$ case representative of thermal scattering. Because of this complication, further discussion of magnetoresistance mobility will be deferred until later. It would be possible, of course, to consider a mobility based on the Corbino magnetoresistance in connection with the constant-τ model. However, it seems better to treat both types of magnetoresistance together (see Section 15*b*).

For the limiting values in strong magnetic fields, the following relationships develop:

$$R_\infty = \lim_{H\to\infty} [1/H\sigma_{xy}(H)] = -1/nec, \qquad \varepsilon > 0 \text{ for electrons.} \tag{12.46}$$

This particular relationship, (12.46), holds not only for spherical energy surfaces as derived here, but under quite general conditions (see discussion in Section 12*b*).

Where it is possible experimentally to reach the strong-magnetic-field plateau, Eq. (12.46) is an extremely useful relation. For example, we can obtain very simply the carrier concentration, without being specifically concerned about the scattering mechanism or the shape of the constant-energy surfaces. The *conductivity* mobility is obtained directly through use of (12.23), namely,

$$\mu(\eta) = R_\infty \sigma(H) c \tag{12.47}$$

and

$$\mu_0(\eta) = R_\infty \sigma_0 c. \tag{12.48}$$

The latter expression was used by Harman *et al.*[183] to obtain mobilities in *p*-type germanium.

Furthermore, by measuring both strong- and weak-field Hall plateaus, one has determined the ratio of Hall and conductivity mobilities for weak fields, that is,

$$\mu_0^H(\eta)/\mu_0(\eta) = R_0/R_\infty. \tag{12.49}$$

The relation between Hall coefficient and carrier density for a single band is often written as[183a]

$$R_0 = -r/nec, \qquad e > 0 \text{ for electrons} \tag{12.50}$$

where, in general, r depends on the scattering mechanism and on the nature of the energy surfaces. The above expression can be generalized to apply for arbitrary values of H and degeneracy, namely,

$$R_H = -r_H(\eta)/nec, \qquad e > 0 \text{ for electrons.} \tag{12.51}$$

The *Hall coefficient factor* $r_H(\eta)$ depends on H, on the reduced Fermi level η (except in the limit of classical statistics), and on the scattering mechanism. In the general case, it also depends on the band structure. Experimentally, however, it is determined by use of the relation

$$R_H/R_\infty = r_H(\eta). \tag{12.52}$$

For the ordinary transverse magnetoresistance we obtain in the strong-magnetic-field limit

$$\Delta\rho/\rho_0 \approx \langle 1/\tau \rangle_{\mathrm{F-D}} \langle \tau \rangle_{\mathrm{F-D}} - 1, \qquad \omega\tau \gg 1 \tag{12.53}$$

and for the Corbino effect,

$$(\Delta\rho/\rho_0)_{E_y=0} \approx \{\omega^2 \langle \tau \rangle_{\mathrm{F-D}} / \langle 1/\tau \rangle_{\mathrm{F-D}}\} - 1, \quad \omega\tau \gg 1 \tag{12.54}$$

$$\approx \{\langle \tau \rangle_{\mathrm{F-D}} / \langle 1/\tau \rangle_{\mathrm{F-D}}\} \omega^2. \tag{12.55}$$

Thus, we see that the magnetoresistance becomes independent of ω, i.e., it *saturates* at high fields. The Corbino magnetoresistance, on the other hand, goes as H^2 at large fields. This is to be expected because of

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[183] T. Harman, R. Willardson, and A. Beer, *Phys. Rev.* **94**, 1065 (1954).

[183a] See, for example, V. A. Johnson and K. Lark-Horovitz, *Phys. Rev.* **79**, 176 (1950).

the loss of the Hall field in the Corbino experiments. Actually, at extremely large values of $\omega\tau$, the Corbino effect is found to deviate from the H^2 curve. Possible causes are influence of minority carriers (Section 17a), inhomogeneities in specimen or magnetic field (Section 27), or orbit quantization effects (Section 28).

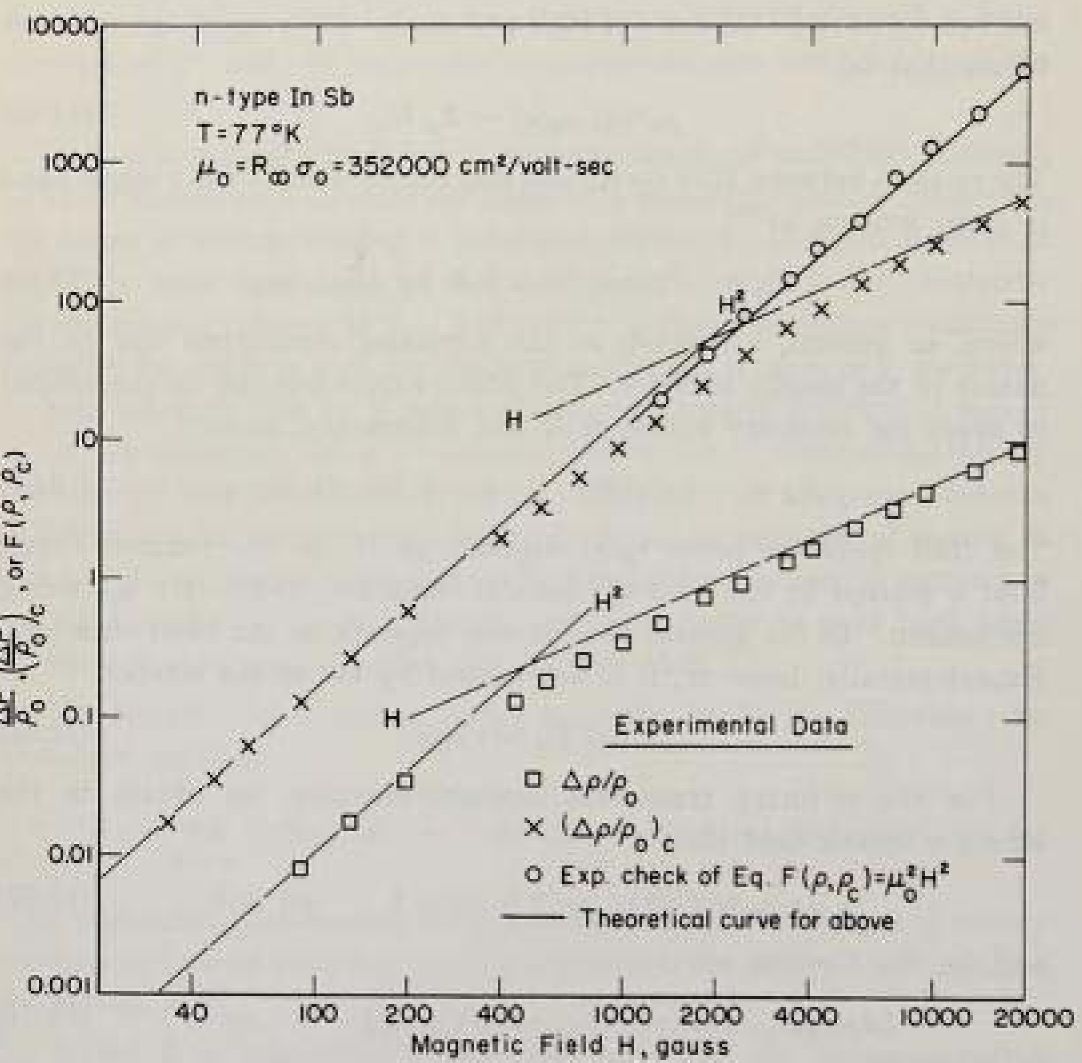

FIG. 10. Magnetic-field dependence of transverse magnetoresistance and Corbino magnetoresistance in InSb specimen. Data are also plotted to provide an experimental check of the theoretical relationship, valid when R_H saturates:

$$\frac{\rho}{\rho_0}\left[\left(\frac{\rho}{\rho_0}\right)_C - \left(\frac{\rho}{\rho_0}\right)\right] \equiv F(\rho, \rho_C) = (\mu_0 H/10^8)^2.$$

The ordinary magnetoresistance was measured on a parallelepiped fabricated from the Corbino disk (after Beer[184]).

One can show quite generally that a deviation in the Corbino magnetoresistance from the H^2 dependence at strong magnetic fields results from a nonsaturation of the ordinary transverse magnetoresistance. This can be seen from the general relation given in Eq. (12.36) which, in the region where the Hall coefficient saturates, can be put in the form

$$\left[\left(\frac{\rho_H}{\rho_0}\right)_{E_y=0} - \left(\frac{\rho_H}{\rho_0}\right)\right]\frac{\rho_H}{\rho_0} = \left(\frac{\mu_0 H}{c}\right)^2, \qquad \frac{\mu_0 H}{c} \gg 1 \qquad (12.55a)$$

where the conductivity mobility is given by Eq. (12.48). If laboratory units (μ in cm^2/volt-sec and H in gauss) are used, then the factor c is replaced by 10^8.

An experimental check on relation (12.55a) is available from some unpublished measurements of Bate.[184] The procedure was to measure first the Corbino effect at 77 °K as a function of magnetic field on a disk with soldered annular electrodes. Since the disk was 12.9 mm in diameter and 2.28 mm thick, with a hole 3.35 mm in diameter at the center, it was subsequently possible to cut a sample in the shape of a rectangular parallelepiped, and then to measure directly the Hall effect, resistivity, and transverse magnetoresistance as functions of field at 77 °K. A determination of the resistance of the contacts and connecting leads of the original Corbino disk was then made by subtracting from its measured resistance that resistance calculated using its dimensions and the resistivity measured on the parallelepiped. The total resistance of the contacts and leads was about 35% of the disk resistance at zero field. The Corbino data were then corrected for contact effects by subtracting the contact resistance (assumed independent of magnetic field) from the resistance of the disk in the field. The resulting data, along with those of the transverse magnetoresistance, are shown in Fig. 10. The fairly good coincidence of the circles and the theoretical line indicates a satisfying verification of the relation between Corbino effect, transverse magnetoresistance, and conductivity mobility.

e. Arbitrary Magnetic Field Strengths

When H is arbitrary, expansions are not possible and the fractional functions of the relaxation times shown in (12.12) and (12.13) must be

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[184] See A. C. Beer, *J. Appl. Phys.* **32**, 2107 (1961).

used. The expression for the conductivity coefficient $\sigma_{xx}(H)$ assumes the following form for the simple power dependence of τ, namely, $\tau = \tau_0 \, x^\lambda$:

$$\sigma_{xx}(H) = -\frac{2}{3}\frac{ne^2}{m^*}\frac{\tau_0}{F_{1/2}(\eta)}\int\limits_0^\infty \frac{x^{\lambda+3/2}}{1+\gamma_\lambda x^{2\lambda}}\frac{\partial f_0}{\partial x}\,dx, \qquad H \equiv H_z \qquad (12.56)$$

where

$$\gamma_\lambda = \omega^2 \tau_0{}^2. \qquad (12.57)$$

It is to be noted that $\partial f_0/\partial x$ is a negative quantity.

It is convenient to express τ_0 in terms of the zero-magnetic-field conductivity mobility $\mu_0(\eta)$, where $\sigma_0 = -ne\mu_0(\eta)$ [cf. Eq. (12.23)]:

$$\mu_0(\eta) = -\frac{2}{3}\frac{e}{m^*}\frac{(\lambda+\frac{3}{2})}{F_{1/2}(\eta)}F_{\lambda+1/2}(\eta)\tau_0, \qquad e > 0 \text{ for electrons.} \qquad (12.58)$$

Thus

$$\sigma_{xx}(H) = ne\mu_0(\eta)[(\lambda+\tfrac{3}{2})F_{\lambda+1/2}(\eta)]^{-1}\int\limits_0^\infty \frac{x^{\lambda+3/2}}{1+\gamma_\lambda x^{2\lambda}}\frac{\partial f_0}{\partial x}\,dx \qquad (12.59)$$

and

$$\gamma_\lambda = \left[\frac{3F_{1/2}(\eta)}{2(\lambda+\frac{3}{2})F_{\lambda+1/2}(\eta)}\right]^2\left[\frac{\mu_0(\eta)H}{c}\right]^2 \qquad (12.60)$$

Similarly, for $\sigma_{xy}(H)$, we obtain[184a]

$$\sigma_{xy}(H) = -ne\mu_0(\eta)\gamma_\lambda{}^{1/2}\,[(\lambda+\tfrac{3}{2})F_{\lambda+1/2}(\eta)]^{-1}\int\limits_0^\infty \frac{x^{2\lambda+3/2}}{1+\gamma_\lambda x^{2\lambda}}\frac{\partial f_0}{\partial x}\,dx,$$

$$H \equiv H_z. \qquad (12.61)$$

[184a] In these expressions the sign conventions are: For electrons: e, $\gamma_\lambda{}^{1/2}$, and $\omega > 0$; $\mu_0 < 0$. For positive carriers e, $\gamma_\lambda{}^{1/2}$, and $\omega < 0$; $\mu_0 > 0$.

13. COMMONLY ENCOUNTERED SCATTERING MECHANISMS — EXACT STATISTICS

Appropriate descriptions of the scattering of charge carriers in solids, even in the cases considered here where the processes can be represented by relaxation times (Section 6), are complicated, and the transport integrals have been evaluated principally for certain specialized cases. We shall consider the effects of several of the more representative scattering mechanisms on the galvanomagnetic effects.

An analysis of the scattering process itself will not be presented. For information on these aspects, the reader is referred to the review articles by Brooks[185] and by Blatt,[186] to Wilson's book,[187] and to literature to be cited later (e.g., Sections 25, 30, and 31). The case of conduction in thin wires and films of metals is included in an article by Sondheimer.[188] We shall discuss the energy dependence of the relaxation time characteristic of the scattering mechanisms, and shall present explicit evaluations of the transport parameters for each $\tau(\varepsilon)$.

a. Scattering by Thermal Lattice Vibrations

At this point, it is desirable to restrict our scope to semimetals and semiconductors. Most of the preceding developments have been sufficiently general and would apply to metals, with $\eta \gg 1$. The interaction between the charge carriers and the lattice vibrations, however, is very complex. It therefore seems desirable to take advantage of the simplifications which result from the lower concentration of charge carriers.

One of the simplest kinds of interactions between carriers and lattice is the scattering by the longitudinal acoustic modes of the lattice vibrations, i.e., by the emission or absorption of an acoustical phonon. In such a case, the relaxation time is given by[188a]

$$\tau_L = \tau_L{}^0 x^{-1/2}. \tag{13.1}$$

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[185] See p. 144 ff. of reference 33.

[186] See p. 287 ff. of reference 44.

[187] See p. 251 ff. of reference 9.

[188] E. H. Sondheimer, *Advances in Phys.* **1**, 1 (1952).

[188a] The designations τ_0 and τ^0 will be used interchangeably in this article to indicate the energy-independent factor in the relaxation time, depending on the particular combinations of subscripts or superscripts.

It will subsequently be shown that the above relation leads to a $T^{-3/2}$ temperature dependence for the conductivity mobility.

Calculations based on (13.1) have been very useful in studying the properties of semiconductors where the temperature and degree of purity were high enough so that the scattering by ionized impurities was not appreciable.[189–191] When fairly good quantitative agreement between theory and experiment is desired, however, it is usually necessary to consider additional mechanisms of interaction between the carriers and the lattice for a number of semiconductors.[186,192] For example, in *p*-type germanium the influence of the optical modes is significant; in semiconductors with band structure described by the many-valley model, the effect of intervalley scattering may be important. Finally, if there are two unlike atoms in a unit cell so that optical vibrations can produce an electric polarization (polar solids), then further complications occur. These more sophisticated considerations will be discussed later in connection with the individual semiconductor, or class of semiconductors, requiring such treatment.

The form of τ_L^0 in (13.1) for acoustical mode scattering has been presented by a number of authors.[187,193,194] The particular result we shall quote here arises most directly from calculations based on the concept of a *deformation potential*.[195,196] This scheme is applicable when strains vary slowly with interatomic distances so that the solid can be treated as a continuum, and local deformations produced by the lattice waves are similar to those in homogeneously deformed crystals. The mobilities of the carriers are related to shifts of the band edges associated with dilatations due to the longitudinal waves. The technique has been extended by Hunter and Nabarro[197] to include inhomogeneously deformed lattices and also the changes in effective masses of the electrons with strain.

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[189] See, for example, the results given in Shockley's book.[32]

[190] G. L. Pearson and J. Bardeen, *Phys. Rev.* **75**, 865 (1949).

[191] P. P. Debye and E. M. Conwell, *Phys. Rev.* **93**, 693 (1954).

[192] E. M. Conwell, *Proc. I.R.E.* **46**, 1281 (1958).

[193] See p. 560 in the article by Sommerfeld and Bethe.[34c]

[194] F. Seitz, *Phys. Rev.* **73**, 549 (1948).

[195] W. Shockley and J. Bardeen, *Phys. Rev.* **77**, 407 (1950).

[196] J. Bardeen and W. Shockley, *Phys. Rev.* **80**, 72 (1950).

[197] S. C. Hunter and F. R. N. Nabarro, *Proc. Roy. Soc.* **A220**, 542 (1953).

The theory is used to analyze effects of edge and screw dislocations in metals, with specifical calculations for copper and sodium.

The deformation potential calculation gives the following result:

$$\tau_L = \tau_L^0 x^{-1/2} \tag{13.2}$$

where

$$\tau_L^0 = \frac{\pi\hbar^4 \rho u_l^2}{\sqrt{2}\, E_1^2 m^{*3/2}(kT)^{3/2}}. \tag{13.3}$$

The parameter ρ is the density of the solid, u_l is the velocity of the longitudinal sound waves, and E_1 is the change in band edge energy per unit dilation. The energy E_1 is related by a numerical factor to the interaction constant C appearing in the earlier theories of lattice mobility.[187, 193, 194] It can be estimated from independent measurements such as changes in energy gap with pressure and with temperature.[186] Since E_1 is essentially constant with temperature, it is seen that τ_L^0 varies as $T^{-3/2}$.

When (13.2) is used to evaluate the coefficients defined in the preceding section, one obtains for the conductivity mobility in the zero-magnetic-field limit,

$$\mu_0^L(\eta) = -\frac{2}{3}\frac{e\tau_L^0}{m^*}\frac{F_0(\eta)}{F_{1/2}(\eta)} = -\frac{\sqrt{2}\,\pi e\hbar^4 \rho u_l^2 F_0(\eta)}{3E_1^2 m^{*5/2}(kT)^{3/2} F_{1/2}(\eta)}, \tag{13.4}$$

and for the corresponding Hall mobility

$$\mu_0^{H,L}(\eta) = -\frac{1}{2}\frac{e\tau_L^0}{m^*}\frac{F_{-1/2}(\eta)}{F_0(\eta)}, \qquad e > 0 \text{ for electrons.} \tag{13.5}$$

Thus, the mobilities have mass and temperature dependencies of $m^{*-5/2}$ and $T^{-3/2}$, respectively, for scattering by acoustic mode lattice vibrations.

The other quantities in the zero-magnetic-field limit are

$$r_0^L(\eta) = \frac{\mu_0^{H,L}(\eta)}{\mu_0^L(\eta)} = \frac{3}{4}\frac{F_{-1/2}(\eta)F_{1/2}(\eta)}{[F_0(\eta)]^2} \tag{13.6}$$

$$\left(\frac{\Delta\rho}{\rho_0}\right)^L \simeq (\omega\tau_L^0)^2\left\{\frac{\mathcal{F}_{-1}(\eta)}{F_0(\eta)} - \frac{1}{4}\left[\frac{F_{-1/2}(\eta)}{F_0(\eta)}\right]^2\right\}, \qquad \omega\tau_L^0 \ll 1 \tag{13.7}$$

where $\mathscr{F}_{-1}(\eta)$ is Dingle's function, Eq. (12.8):

$$\mathscr{F}_{-1}(\eta) = (e^{-\eta} + 1)^{-1}. \tag{13.8}$$

The Corbino magnetoresistance is given by

$$(\Delta\rho/\rho_0)^L_{E_y=0} \cong (\omega\tau_L{}^0)^2 \mathscr{F}_{-1}(\eta)/F_0(\eta), \qquad \omega\tau_L{}^0 \ll 1. \tag{13.9}$$

Both (13.7) and (13.9) can be expressed in terms of the conductivity mobility, since with the use of (13.4) one obtains[184a]

$$\omega\tau_L{}^0 = -\frac{3}{2}\frac{F_{1/2}(\eta)}{F_0(\eta)}\frac{\mu_0{}^L H}{c}. \tag{13.10}$$

In the limit of strong magnetic fields, the corresponding expressions are

$$(\Delta\rho/\rho_0)^L \approx (8/9)\{F_0(\eta)F_1(\eta)/[F_{1/2}(\eta)]^2\} - 1, \quad \omega\tau_L \gg 1, \tag{13.10a}$$

$$(\Delta\rho/\rho_0)^L_{E_y=0} \approx \omega^2\tau_L{}^{02} F_0(\eta)/2F_1(\eta), \qquad \omega\tau_L \gg 1 \tag{13.11}$$

$$\approx \frac{9}{8}\frac{F^2_{1/2}(\eta)}{F_0(\eta)F_1(\eta)}\left(\frac{\mu_0^L H}{c}\right)^2, \qquad \omega\tau_L \gg 1. \tag{13.11a}$$

For the case of arbitrary magnetic field strengths, the expressions (12.59 – 12.61) must be utilized. With $\lambda = -\frac{1}{2}$ for lattice scattering, we obtain

$$\sigma^L_{xx}(H) = ne\mu_0{}^L(\eta)\,\frac{1}{F_0(\eta)}\int_0^\infty \frac{x^2}{\gamma^L + x}\frac{\partial f_0}{\partial x}\,dx, \qquad e > 0 \text{ for electrons,} \tag{13.12}$$

$$\sigma^L_{xy}(H) = -\,ne\mu_0{}^L(\eta)\,\frac{(\gamma^L)^{1/2}}{F_0(\eta)}\int_0^\infty \frac{x^{3/2}}{\gamma^L + x}\frac{\partial f_0}{\partial x}\,dx \tag{13.12a}$$

where

$$\gamma^L = \left[\frac{3F_{1/2}(\eta)}{2F_0(\eta)}\right]^2\left[\frac{\mu_0{}^L(\eta)H}{c}\right]^2. \tag{13.12b}$$

It is to be noted that in the present sign convention $\mu_0{}^L(\eta)$ is negative for transport by electrons.[184a]

b. Scattering by Ionized Impurities

The usual treatments of the scattering of carriers by an array of ionized centers lead to a relaxation time given by an expression of the following type:[198]

$$\tau_I = \frac{K^2(2m^*)^{1/2}(kT)^{3/2}}{\pi e^4 N_I} \frac{x^{3/2}}{g(n^*, T, x)} = \tau_I^0 \frac{x^{3/2}}{g(n^*, T, x)} \tag{13.13}$$

where x is the reduced energy ($\varepsilon \equiv xkT$), $g(n^*, T, x)$ is a slowly varying function, K is the dielectric constant, N_I is the total density of ionized impurities, and n^* depends in general on both N_I and the carrier concentration, i.e., on the degree of compensation.

The problem was originally treated by Conwell and Weisskopf,[199] who considered the Rutherford scattering of each ion independently. The divergence arising from the increasing contributions at small angle scattering was removed by arbitrarily cutting off the scattering cross sections at an angle corresponding to a closest approach of half the average distance between impurities. The Conwell-Weisskopf theory gives the following result for $g(n^*, T, x)$:

$$g(n^*, T, x) = \ln\,[1 + (KkT/e^2 N_I^{1/3})^2\, x^2]. \tag{13.14}$$

In subsequent work by Brooks and Herring[200] and by Dingle,[201] a screened Coulomb potential was used. This screening, which takes care of the divergence difficulty, arises from the fact that charge carriers distribute themselves around the impurity and cancel its field at large distances. Let us designate this screening distance, or Debye-Hückel length, by a, defined in terms of the screened scattering potential

$$|V(r)| = (e/Kr)e^{-r/a}. \tag{13.15}$$

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[198] A factor of 2 appears inadvertently to have entered the numerator of this equation given in the articles by Brooks[185] and by Blatt[186]. The result quoted here is in agreement with the results of other authors to be cited later. After averaging over energy, (13.13) yields the same $\mu_0{}^I$ as is quoted in all the literature examined.

[199] E. M. Conwell and V. F. Weisskopf, *Phys. Rev.* 77, 388 (1950).

[200] See H. Brooks, *Phys. Rev.* 83, 879 (1951); see also references 185 and 191.

[201] R. B. Dingle, *Phil. Mag.* [7] 46, 831 (1955).

The slowly varying function g in Eq. (13.13) can then be written[201, 202]

$$g(n^*, T, x) = \ln(1 + z) - z/(1 + z), \qquad z \equiv (2ka)^2 \tag{13.16}$$

where k is the magnitude of the wave vector of the charge carrier; and the screening distance "a" is a function of n^*, T, and x. It is important to point out that assumptions inherent in the classical derivation, that is, (13.14), require essentially that

$$ka \gg 1. \tag{13.17}$$

The quantum-mechanical treatment, which leads to (13.16), is based on the Born approximation, and it is pertinent to note that in this range of validity, (13.17) is satisfied for representative values of the parameters.[186, 202] Hence we may state the range of validity of the treatments discussed thus far to be given by

$$z \gg 1. \tag{13.18}$$

Other considerations pertinent to the range of applicability of conventional scattering theory are discussed in Section 31.

Consideration of the elastic displacements of the lattice as a result of polarization of atoms surrounding the impurity ion was done by Horie.[203] His results give an equation identical to (13.13) except that the slowly varying factor $g(n^*, T, x)$ is more complicated than is the function in (13.14) or (13.16).

When only impurities of one sign are present in the crystal, the screening distance is the same order as the mean distance between impurities and the Conwell-Weisskopf and Brooks-Herring, Dingle treatments give closely the same results. Where compensation occurs, however, and the charge-carrier density is less than the ionized impurity density, the latter formula can give a lower mobility for the same N_I, because of the reduction in screening for smaller n.[191]

By using the results obtained by Dingle, we can write an explicit expression for z, which for a single type of charge carrier can be put in the form

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[202] N. Sclar, *Phys. Rev.* **104**, 1548 (1956).

[203] C. Horie, *Sci. Repts. Tôhoku Univ.* **34**, 29 (1950).

$$z = \frac{4Km^*(kT)^2}{\pi e^2 \hbar^2 n}\left[\frac{F_{1/2}(\eta)}{F_{-1/2}(\eta)}\right] x \tag{13.19}$$

where n is the density of charge carriers in the conduction band, Eq. (12.6). Where both electrons and holes are present, the reference[201] should be consulted.

In the limit of classical statistics, $\eta \ll 0$, relation (13.19) yields the Brooks-Herring expression if n is replaced by n^*. The density n^* can differ slightly from the carrier density n when compensation occurs.[185, 186] The explicit expression will be given in the section dealing with classical statistics.

Evaluation of the transport coefficients presents somewhat of a problem since the integrals containing (13.13) cannot be evaluated analytically. However, in view of the requirement that $z \gg 1$, it is seen from (13.16) and (13.19) that $g(n^*, T, x)$ is a slowly varying function of x, so that fairly good approximations are possible. The procedure is to evaluate the function $g(n^*, T, x)$ at an appropriately chosen value of x, designated by $\bar{x}$, and thus to take it outside the integral. One thereby obtains

$$\langle \tau_I^s \rangle_{F-D} \cong -\frac{2}{3}\frac{1}{F_{1/2}(\eta)}\left[\frac{\tau_I^0}{g(n^*, T, \bar{x})}\right]^s \int_0^\infty x^{3(s+1)/2}\frac{\partial f_0}{\partial x}\,dx. \tag{13.20}$$

The value of $\bar{x}$ is commonly determined by the condition that the integrand remaining after the removal of $g(n^*, T, \bar{x})$ be a maximum at $x = \bar{x}$. This technique was used in the original calculations by Conwell and Weisskopf. An alternative procedure, suggested by Dingle,[201] is to choose $\bar{x}$ as that value which, when used as an upper limit cutoff for the integral remaining after removal of $g(n^*, T, x)$, yields a median value. Since the first method of determining $\bar{x}$ appears most consistently in the literature, it will be considered further. It would, however, be interesting to have a quantitative comparison of the two techniques.

For determining transport coefficients in the limiting cases of weak and of strong magnetic fields — and because of the complicated integrals these are the only examples which will be considered in the case where an arbitrary degree of degeneracy exists — it is necessary to know averages of different powers of τ_I. These are given in (13.20), where $\bar{x}$ is determined by the condition

$$[x^{3(s+1)/2}(\partial f_0/\partial x)]_{x=\bar{x}} = \text{maximum}. \tag{13.21}$$

From the above, one obtains

$$[\bar{x} - 3(s+1)/2] = [\bar{x} + 3(s+1)/2] \exp(\eta - \bar{x}). \tag{13.22}$$

It is important to note that $\bar{x}$ depends on s and on η (except in the classical limit). The significance of this point has been brought out by Mansfield.[204, 205]

Carrying out the integration in (13.20) gives

$$\langle \tau_I^s \rangle_{\mathrm{F-D}} \cong \frac{2}{3}\left[\frac{3s+3}{2}\right] \frac{F_{(3s+1)/2}(\eta)}{F_{1/2}(\eta)} \left[\frac{\tau_I^0}{g(\bar{x}_{s,\eta})}\right]^s \tag{13.23}$$

where $g(n^*, T, \bar{x}_{s,\eta})$ is abbreviated by $g(\bar{x}_{s,\eta})$ with $\bar{x}_{s,\eta}$ given by the solution of (13.22) for the appropriate values of s and η. The quantity τ_I^0 is defined in (13.13).

The mobilities and galvanomagnetic coefficients representative of ionized impurity scattering are then given by

$$\mu_0^I(\eta) = -\frac{2F_2(\eta)}{F_{1/2}(\eta)} \frac{e\tau_I^0}{m^* g(\bar{x}_{1,\eta})} = -\frac{2F_2(\eta)}{F_{1/2}(\eta)} \left\{\frac{2^{1/2} K^2 (kT)^{3/2}}{\pi e^3 m^{*1/2} N_I g(\bar{x}_{1,\eta})}\right\}, \tag{13.24}$$

$$\mu_0^{H,I}(\eta) = -\frac{3}{2} \frac{F_{7/2}(\eta) g^2(\bar{x}_{1,\eta})}{F_2(\eta) g^2(\bar{x}_{2,\eta})} \left\{\frac{2^{1/2} K^2 (kT)^{3/2}}{\pi e^3 m^{*1/2} N_I g(\bar{x}_{1,\eta})}\right\}, \tag{13.25}$$

$$r_0^I(\eta) = \frac{\mu_0^{H,I}(\eta)}{\mu_0^I(\eta)} = \frac{3}{4} \frac{F_{7/2}(\eta) F_{1/2}(\eta) g^2(\bar{x}_{1,\eta})}{F_2^2(\eta) g^2(\bar{x}_{2,\eta})}, \tag{13.26}$$

$$\left(\frac{\Delta\rho}{\rho_0}\right)^I \cong \left\{\frac{2F_5(\eta) g^3(\bar{x}_{1,\eta})}{F_2(\eta) g^3(\bar{x}_{3,\eta})} - \left[\frac{3F_{7/2}(\eta) g^2(\bar{x}_{1,\eta})}{2F_2(\eta) g^2(\bar{x}_{2,\eta})}\right]^2\right\} \left[\frac{\omega\tau_I^0}{g(\bar{x}_{1,\eta})}\right]^2, \quad \frac{\omega\tau_I^0}{g} \ll 1 \tag{13.27}$$

$$\left(\frac{\Delta\rho}{\rho_0}\right)^I_{E_y=0} \cong \frac{2F_5(\eta) g^3(\bar{x}_{1,\eta})}{F_2(\eta) g^3(\bar{x}_{3,\eta})} \left[\frac{\omega\tau_I^0}{g(\bar{x}_{1,\eta})}\right]^2, \quad \frac{\omega\tau_I^0}{g} \ll 1. \tag{13.28}$$

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[204] R. Mansfield, *Proc. Phys. Soc. (London)* **B69**, 76 (1956).
[205] R. Mansfield, *Proc. Phys. Soc. (London)* **B69**, 862 (1956).

In the limit of strong magnetic fields, one obtains

$$\left(\frac{\Delta\rho}{\rho_0}\right)^I \approx \frac{4\mathscr{F}_{-1}(\eta)F_2(\eta)g(x^*_{-1,\eta})}{3[F_{1/2}(\eta)]^2 g(\bar{x}_{1,\eta})} - 1, \qquad \frac{\omega\tau_I^0}{g} \gg 1, \tag{13.29}$$

$$\left(\frac{\Delta\rho}{\rho_0}\right)^I_{E_y=0} \approx \frac{3F_2(\eta)g(\bar{x}_{1,\eta})}{\mathscr{F}_{-1}(\eta)g(x^*_{-1,\eta})}\,[\omega\tau_I^0/g(\bar{x}_{1,\eta})]^2, \qquad \frac{\omega\tau_I^0}{g} \gg 1. \tag{13.30}$$

The parameter x^*_{-1} is introduced inasmuch as the treatment culminating in Eq. (13.22) breaks down when $s = -1$, since in such cases the integrand with $g(x)$ constant no longer possess a maximum. The designation $\bar{x}_{-1}$, therefore, has no meaning. The x^*_{-1} can, however, be determined by the method of Dingle [see discussion following (13.20)], or the complete integrand including the $g(x)$ can be handled by numerical methods.

The factor $\omega\tau_I^0/g$ can be expressed in terms of the conductivity mobility at zero magnetic field by means of (13.24), thus

$$\frac{\omega\tau_I^0}{g(\bar{x}_{1,\eta})} = -\left[\frac{F_{1/2}(\eta)}{2F_2(\eta)}\right]\left[\frac{\mu_0{}^I(\eta)H}{c}\right], \qquad \mu_0{}^I(\eta) < 0 \text{ for electrons.} \tag{13.31}$$

Several investigators have made studies of the range of validity of the simplified treatment of the impurity scattering problem presented in this section. Inasmuch as explicit results were obtained only for the case of classical statistics, detailed discussion will be deferred until later. We shall point out here, however, that it is generally desirable to take account of the behavior of $g(n^*, T, \bar{x}_{s,\eta})$ when plotting the temperature dependence of $\mu_0{}^I(\eta)$ and $\mu_0{}^{H,I}(\eta)$. Also in calculating $r_0{}^I(\eta)$ for relating the Hall coefficient and carrier density, it is best to determine the values of $g(\bar{x}_2, \eta)$ and of $g(\bar{x}_1, \eta)$, rather than to replace their ratio by unity.[204, 205] In any case, it appears quite essential that the theory be applied only for those cases where $z \gg 1$, as specified in (13.18).

The general expressions (12.59) - (12.61) for the case of arbitrary magnetic field strengths, when used with the relaxation times given in this section, yield formally:

$$\sigma^I_{xx}(H) = ne\mu_0{}^I(\eta)\,\frac{g(\bar{x}_{1,\eta})}{3F_2(\eta)}\int\limits_0^\infty \frac{x^3}{1 + [\omega\tau_I^0/g(x)]^2 x^3}\,\frac{1}{g(x)}\,\frac{\partial f_0}{\partial x}\,dx, \tag{13.32}$$

$$\sigma^{I}_{xy}(H) = ne\mu_0{}^I(\eta)\,\frac{F_{1/2}(\eta)g^2(\bar{x}_{1,\eta})}{6F_2{}^2(\eta)}\,\frac{\mu_0{}^I H}{c}\int\limits_0^\infty \frac{x^{9/2}}{1 + [\omega\tau_I{}^0/g(x)]^2\,x^3}\,\frac{1}{g^2(x)}\,\frac{\partial f_0}{\partial x}\,dx \tag{13.33}$$

where $e > 0$ and $\mu_0{}'(\eta) < 0$, for transport by electrons. The value of $\bar{x}_{1,\eta}$ is given from (13.22) with $s = 1$; the $g(x)$ is, of course, the function written as $g(n^*, T, x)$ in (13.13).

In the above equations, $g(x)$ now also occurs in the binomial denominator. Therefore, the method for obtaining an approximate value of the integral by replacing $g(x)$ by a constant $g(\bar{x})$, is not so simple as in the case of (13.20). The value of $\bar{x}$ — that is, the value of x which causes the integrand with the $g(x)$ replaced by $g(\bar{x})$ to be a maximum — is now a function of $\omega\tau_I{}^0/g(\bar{x})$ as well as of η. For arbitrary values of η, there is not an overwhelming advantage in using this procedure, inasmuch as a numerical integration is still necessary after the $g(x)$ is replaced by $g(\bar{x})$. Therefore, one might as well work directly with the integrand containing the $g(x)$. This is obviously true if results are required for only one value of n^* and of T [see Eq. (13.16) and following] so that the other parameters in $g(x)$ are constant. Of course, if a number of values of n^* and T are to be used, the $g(\bar{x})$ technique can save repeated numerical integrations. With classical statistics, on the other hand, the simpler integral can be evaluated in terms of tabulated functions, and numerical integrations are avoided.

The simplest approximation is to neglect the difference between $g(\bar{x}_{1,\eta,\omega})$ and $g(\bar{x}_{1,\eta})$. In such a case the g's drop out of (13.32) except for the magnetic field parameter, which is then given by (13.31). In (13.33), of course, the factor $[g(\bar{x}_{1,\eta})/g(\bar{x}_{2,\eta})]^2$ will enter. No information is available to the error introduced either by this approximation or by the procedure involving determination of the $\bar{x}$ appropriate to the particular value of $\omega\tau_I{}^0/g(\bar{x})$.

c. Mixed Scattering

A more realistic treatment of the relaxation process involves consideration of two or more scattering mechanisms, the effectiveness of each being dependent on the energy of the charge carrier.[178] This requirement becomes especially stringent when galvanomagnetic data are studied over a wide range of temperatures. For the two scattering mechanisms discussed in the preceding sections, it is seen that at high temperatures the relaxation time is limited principally by interaction of the charge carriers with the lattice vibrations; at low temperatures, by the interaction with ionized impurities. Inclusion of these particular processes in the transport integrals provides a composite scattering system which has been studied frequently in the literature.

Since $1/\tau_i$ is proportional to the probability of scattering by mechanism i, one obtains for the case of mixed scattering by lattice vibrations and ionized impurities

$$1/\tau = 1/\tau_L + 1/\tau_I. \tag{13.34}$$

This representation assumes, of course, that the two scattering processes can be regarded as independent of each other. For a further discussion of this point, consult Section 30*b*.

Hence, from (13.2) and (13.13),

$$\tau = \frac{\tau_L{}^0\, x^{3/2}}{(\tau_L{}^0/\tau_I{}^0)g(x) + x^2} \tag{13.35}$$

where g is the slowly varying function of n, T, and x discussed previously.

Because of the complexity of the integrals, the application of (13.35) is usually restricted to the limiting cases of weak or strong magnetic fields or to where classical statistics are applicable. The transport coefficients in the former case will depend on averages of powers of τ, thus:

$$\langle \tau^s \rangle_{\mathrm{F-D}} = -\frac{2}{3}\frac{(\tau_L{}^0)^s}{F_{1/2}(\eta)}\int_0^\infty \frac{x^{3(s+1)/2}}{[x^2 + (\tau_L{}^0/\tau_I{}^0)g(x)]^s}\frac{\partial f_0}{\partial x}\,dx. \tag{13.36}$$

To simplify the above integral by the $g(\bar{x})$ technique requires that $\bar{x}$ be chosen so that

$$\frac{x^{3(s+1)/2}}{(x^2+\beta)^s}\frac{e^{x-\eta}}{(1+e^{x-\eta})^2} = \text{maximum} \tag{13.37}$$

where

$$\beta \equiv (\tau_L{}^0/\tau_I{}^0)g(\bar{x}_{s,\eta,\beta}). \tag{13.38}$$

The solution for (13.37) is

$$\frac{(3-s)\bar{x}^2 + 3(s+1)\beta}{2(\bar{x}^3 + \beta\bar{x})} = \frac{e^{\bar{x}-\eta} - 1}{e^{\bar{x}-\eta} + 1} = \tanh\frac{\bar{x}-\eta}{2} \tag{13.39}$$

where $\bar{x} \equiv \bar{x}_{s,\eta,\beta}$. A relation such as the above, for $s = 1$, has been given by Mansfield.[204]

It is seen that x is a function of s, η, and β. For $\beta \to \infty$, (13.39) reduces to (13.22) as it should, since then τ is determined by the ionized impurity scattering. For arbitrary values of η, if results are required only for a single value of n^* and T, it is probably expedient to do the numerical integration on (13.36) directly, as was suggested in connection with (13.32).

It is often convenient to express the "actual" mobility $\mu_0(\eta)$ when mixed scattering occurs in terms of the mobility $\mu_0{}^L(\eta)$, due to lattice scattering and the mobility $\mu_0{}^I(\eta)$ due to ionized impurities.

With the use of (12.25), (13.4), (13.24), (13.36), and (13.38) it is seen that

$$\mu_0(\eta) = \frac{\mu_0{}^L(\eta)}{F_0(\eta)} \int_0^\infty \frac{x^3}{x^2 + \beta} \frac{e^{x-\eta}}{(1 + e^{x-\eta})^2}\, dx \tag{13.40}$$

where

$$\beta = 3\frac{F_2(\eta)}{F_0(\eta)} \frac{\mu_0{}^L(\eta)}{\mu_0{}^I(\eta)} \frac{g(\bar{x}_{1,\eta,\beta})}{g(\bar{x}_{1,\eta,\infty})}. \tag{13.41}$$

Numerical integrations were done on Eq. (13.40) for a sequence of values of η, and results are presented in Fig. 11.

In the classical case, $\eta < -4$, the integral can be evaluated exactly in terms of sine and cosine integrals, and the results have been plotted by Conwell.[206] In the case of extreme degeneracy, the integrand is large only for energies near the Fermi level, and hence the mobilities combine as do the relaxation times; that is,

$$1/\mu_0 = 1/\mu_0{}^L + 1/\mu_0{}^I, \qquad \eta \gg 0. \tag{13.42}$$

It is seen from Fig. 11 that this relation is approximated quite closely when $\eta > 10$. It is also evident that the use of (13.42) for semiconductors,

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[206] E. M. Conwell, *Proc. I.R.E.* 40, 1327 (1952).

as was done in some of the early literature, can be a rather rough approximation when the two mobilities are nearly equal.

A numerical evaluation of (13.40) has also been carried out by Mansfield.[204] His results are presented in a slightly different form, which involves sums of partial resistivities. It is to be noted that Mansfield's

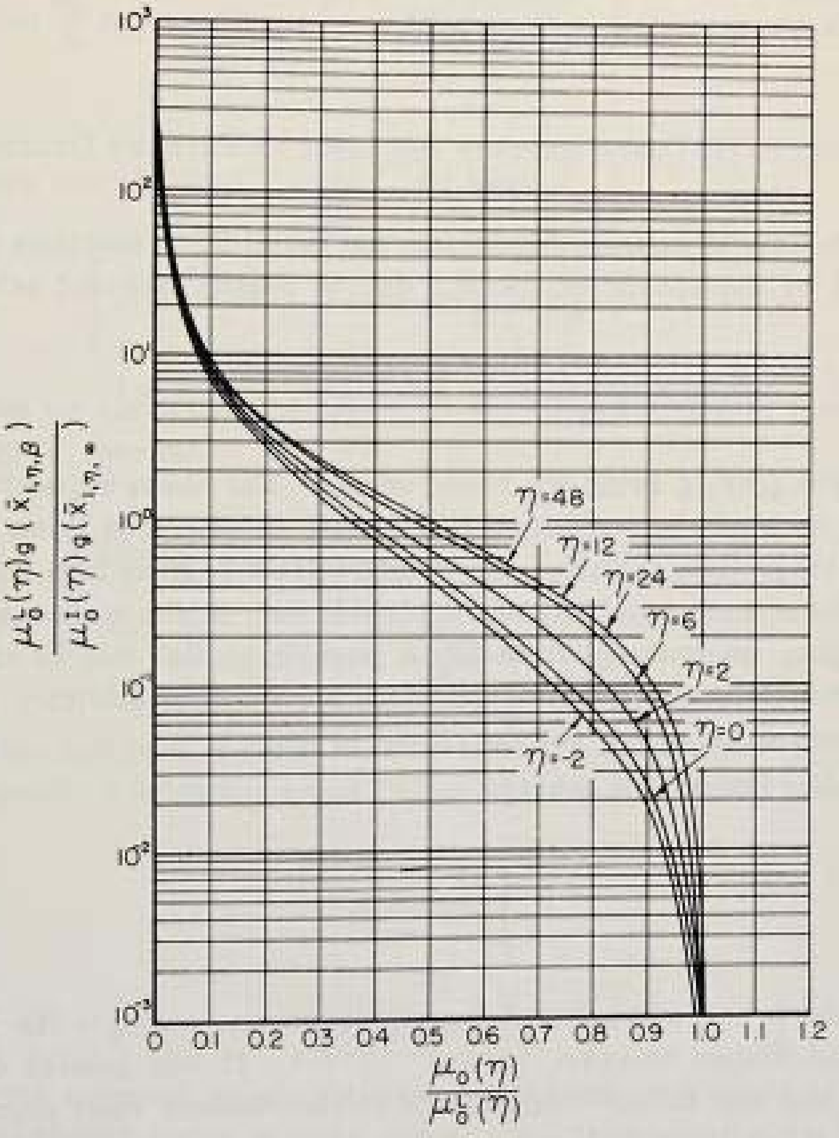

FIG. 11. Composition of mobilities for mixed scattering. The actual or resultant conductivity mobility, at weak-magnetic-field strengths, is denoted by $\mu_0(\eta)$. The mobility due to scattering by lattice vibrations is $\mu_0^L(\eta)$; that due to scattering by ionized impurities is $\mu_0^I(\eta)$. The slowly varying functions $g(\bar{x})$ are discussed in the text.

σ_I is equivalent to our σ_0' multiplied by the factor $g(\bar{x}_{1,\eta,\infty})/g(\bar{x}_{1,\eta,\beta})$. Solutions of (13.39), for $s = 1$, are also presented in graphical form so that $\bar{x}_{1,\eta,\beta}$ is readily determined.

Other conductivity coefficients can be expressed through the use of Eqs. (13.36)–(13.39) for higher values of s. The general form of the relationship will not be given here; instead, the results for classical statistics are presented in Section 15.

14. CONDUCTIVITY COEFFICIENTS IN THE LIMIT OF EXTREME DEGENERACY

In the case of extreme degeneracy, the Fermi-Dirac functions can be replaced by asymptotic expressions due to Sommerfeld and others[207]:

$$F_k(\eta) \approx \frac{\eta^{k+1}}{k+1} + \frac{\pi^2}{6} k\eta^{k-1} + \ldots, \qquad \eta \gg 1 \tag{14.1}$$

where $\eta = \zeta/kT$, ζ being the Fermi energy. The above expansion can be used with all the coefficients which were developed in terms of the $F_k(\eta)$ in the preceding sections. The results will not be given here, however, since most of those equations were applicable only in the weak-magnetic-field region. Inasmuch as it readily is possible in the case of extreme degeneracy to develop relationships which are valid for arbitrary values of H, such development will be presented instead.

Consider integrals of the type

$$\int_0^\infty G(\varepsilon) \frac{\partial f_0}{\partial \varepsilon} \partial \varepsilon \tag{14.2}$$

where $G(\varepsilon)$ is any well-behaved function of energy, and f_0 is the Fermi-Dirac distribution function, $[1 + e^{(\varepsilon - \zeta)/kT}]^{-1}$. It was pointed out by Sommerfeld and Bethe[208] that as the carriers become more degenerate — that is, at larger values of ζ/kT — the derivative $\partial f_0/\partial \varepsilon$ becomes

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[207] See, for example, E. C. Stoner, *Phil. Mag.* [7] **21**, 145 (1936). The result is given in reference 166, except for a typographical error in the Eq. (5.3).

[208] See p. 344 of reference 34c.

increasingly larger and sharper in the neighborhood of $\varepsilon = \zeta$. In the limiting case of $T \to 0$, it approaches a δ-function. A good approximation to (14.2) is, therefore, obtained by expanding $G(\varepsilon)$ about $\varepsilon = \eta$. Details have been presented by Seitz[209] and more recently in other works.[210] The result to second-order terms, is

$$\int_0^\infty G(\varepsilon)\frac{\partial f_0}{\partial \varepsilon}\,d\varepsilon = -G(\zeta) - \frac{\pi^2}{6}(kT)^2\left(\frac{d^2G(\varepsilon)}{d\varepsilon^2}\right)_{\varepsilon=\zeta} - \cdots, \qquad \zeta/kT \gg 1 \tag{14.3}$$

or, since $\varepsilon = xkT$ and $\zeta = \eta kT$, we have

$$\int_0^\infty G(x)\frac{\partial f_0}{\partial x}\,dx = -G(\eta) - \frac{\pi^2}{6}\left(\frac{d^2G(x)}{dx^2}\right)_{x=\eta} - \cdots. \tag{14.4}$$

With the use of the *first* term in the above approximation, Eqs. (12.12) and (12.13) become

$$\sigma_{xx}(H) = \frac{ne^2}{m^*}\frac{\tau(\eta)}{1+\omega^2\tau^2(\eta)}, \qquad \eta \gg 1, \quad H \equiv H_z, \tag{14.5}$$

$$\sigma_{xy}(H) = -\frac{ne^2\omega}{m^*}\frac{\tau^2(\eta)}{1+\omega^2\tau^2(\eta)} \tag{14.6}$$

where the first term of (14.1) has been used to approximate $F_{1/2}(\eta)$. From these results it is readily found [using Eqs. (12.26), (12.27), and (12.51)] that

$$\lim_{\eta\to\infty}\,[r_H(\eta)] = 1, \tag{14.7}$$

$$\lim_{\eta\to\infty}\frac{\Delta\rho}{\rho_0} = 0. \tag{14.8}$$

Thus, for spherical energy surfaces and isotropic relaxation times, the Hall coefficient factor is unity and the magnetoresistance vanishes, in

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[209] See p. 147 *in* F. Seitz, "The Modern Theory of Solids." McGraw-Hill, New York, 1940.

[210] See p. 13 of reference 9; p. 205 of reference 44.

the limit of extreme degeneracy, regardless of the magnetic field strength and of the dependence of τ on energy.

Actually, the relations (14.5)–(14.8) are obvious from physical considerations. In the degenerate limit, only those charge carriers having energies of the Fermi surface contribute to transport. Therefore, insofar as the integrals are concerned, the situation is one of constant τ, with $\tau = \tau(\zeta)$.

To obtain a nonzero magnetoresistance, we must consider higher order terms in (14.3), that is derivatives of τ with respect to energy. From (12.12), (12.13), and (12.27) it is readily shown that the transverse magnetoresistance can be written in the form

$$\Delta\rho/\rho_H \equiv 1 - \sigma_H/\sigma_0 = E\omega^2 \tag{14.9}$$

or

$$\Delta\rho/\rho_0 \equiv (\sigma_0/\sigma_H) - 1 = E\omega^2/[1 - E\omega^2] \tag{14.10}$$

where

$$E \equiv \frac{\langle\tau^3[1+\omega^2\tau^2]^{-1}\rangle_{\mathrm{F-D}}\langle\tau[1+\omega^2\tau^2]^{-1}\rangle_{\mathrm{F-D}} - \langle\tau^2[1+\omega^2\tau^2]^{-1}\rangle^2_{\mathrm{F-D}}}{\langle\tau\rangle_{\mathrm{F-D}}\langle\tau[1+\omega^2\tau^2]^{-1}\rangle_{\mathrm{F-D}}}. \tag{14.11}$$

In the above development, use was made of the identity

$$\langle\tau[1+\omega^2\tau^2]^{-1}\rangle_{\mathrm{F-D}} \equiv \langle\tau\rangle_{\mathrm{F-D}} - \omega^2\langle\tau^3[1+\omega^2\tau^2]^{-1}\rangle_{\mathrm{F-D}}. \tag{14.12}$$

As we saw before, to obtain a nonvanishing E, we must include the second term of the approximation (14.4). The algebra, which is somewhat tedious, is simplified by a substitution such as

$$z = \tau x^{3/2}(1+\omega^2\tau^2)^{-1}. \tag{14.13}$$

Also from (12.11), (14.1), and (14.4) it is seen that

$$\langle q\rangle_{\mathrm{F-D}} \cong q(\eta) + \frac{\pi^2}{6}\eta^{-3/2}\left[\frac{d^2(qx^{3/2})}{dx^2}\right]_{x=\eta} + \cdots, \qquad \eta \gg 1. \tag{14.14}$$

The expression for E, to terms in τ', is found to be

$$E = \frac{\pi^2}{3}\frac{\tau'^2}{1+\omega^2\tau^2}, \qquad \eta \gg 1 \tag{14.15}$$

where $\tau' \equiv d\tau/dx$. Hence

$$\frac{\Delta\rho}{\rho_H} = \frac{\pi^2}{3}\frac{(kT)^2\,\omega^2}{1+\omega^2\tau^2}\left(\frac{d\tau}{d\varepsilon}\right)^2_{\varepsilon=\zeta}, \qquad \eta \gg 1, \tag{14.16}$$

$$\frac{\Delta\rho}{\rho_0} = \frac{\pi^2}{3}\frac{(kT)^2\,\omega^2}{1+\omega^2\tau^2}\left(\frac{d\tau}{d\varepsilon}\right)^2_{\varepsilon=\zeta}\Bigg/\left[1-\frac{\pi^2}{3}\frac{(kT)^2\,\omega^2}{1+\omega^2\tau^2}\left(\frac{d\tau}{d\varepsilon}\right)^2_{\varepsilon=\zeta}\right], \qquad \eta \gg 1. \tag{14.17}$$

It is sometimes customary to express results in terms of a mean free path $l(=v\tau)$, as has been done in discussions of (14.16).[31a]

15. Conductivity Coefficients in the Limit of Classical Statistics

When the temperature is relatively high and the charge-carrier density sufficiently low so that there are a number of unoccupied energy states at low energies, then Maxwell-Boltzmann or classical statistics are applicable. In such cases, $\partial f_0/\partial\varepsilon$ looses its sharpness at $\varepsilon = \zeta$,[211] and contributions to the transport integrals occur throughout a wide range of ε. In these cases, $\eta \ll 0$, so that

$$f_0 \cong e^{\eta}\,e^{-x}, \qquad \partial f_0/\partial x \cong -e^{\eta}\,e^{-x}, \qquad \eta \ll 0 \tag{15.1}$$

and it is readily established that[166]

$$F_k(\eta) \cong k!\cdot e^{\eta}, \qquad \eta \ll 0. \tag{15.2}$$

It also follows from (12.6) that

$$n = \frac{1}{4\pi^3}\left(\frac{2\pi m^*\,kT}{\hbar^2}\right)^{3/2} e^{\eta}. \tag{15.3}$$

All of the relationships developed in Sections 12–14 are readily applied to the present case through the use of (15.2). However, because of the simpler form of $\partial f_0/\partial x$ for classical statistics, many of the developments have been carried further. These will be discussed subsequently. The average of a quantity over the classical distribution, cf. (12.11), is given by

$$\langle q\rangle \equiv \frac{4}{3\sqrt{\pi}}\int_0^\infty q(x)\,x^{3/2}\,e^{-x}\,dx. \tag{15.4}$$

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[211] See for example, Fig. 2, p. 145 of reference 209.